Differentiation

Function :

$$y = f(x)$$

Here, y is the function of x.

means if we change the value of x, the value of y also changes and for every value of x we will get a single value of y

eg: $y = 3x+1$

If $x = 1$, $y = 4$

If $x = 2$, $y = 7$

- $\frac{\Delta y}{\Delta x}$: Average rate of change of y with respect to x.

- If $\Delta x \to 0$: If delta x tends to 0.

  $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

  $= \frac{dy}{dx}$

- $\frac{dy}{dx}$ (read as) : a) differentiation of y with respect to x
  b) 1st derivative of y with respect to x
  c) Instantaneous rate of change of y with respect to x.
  d) Slope of (y-x) graph.

example of differentiation:

i) velocity (instantaneous)

$v = \frac{ds}{dt}$ where s = displacement

t = time take and

v = Instantaneous velocity

ii) Acceleration (instantaneous)

$a = \cancel{dp}\ \frac{dv}{dt}$ where v = velocity

t = time taken and

a = Instantaneous acceleration.

iii) Force (~~ind~~ instantaneous)

$F = \frac{dp}{dt}$ where p = momentum

t = time ~~p~~taken and

F = Instantaneous force.

Basic formulae of differentiation.

1 If y = constant

$\frac{dy}{dx} = \frac{d(\text{constant})}{dx} = 0$

eg: • If $y = 50$, then $\frac{dy}{dx} = \frac{d(50)}{dx} = 0$.

• If $y = 2012$, then $\frac{dy}{dx} = \frac{d(2012)}{dx} = 0$

• If $y = \pi$, then $\frac{dy}{dx} = \frac{d(3.14)}{dx} = 0$

• If $y = e$, then $\frac{dy}{dx} = \frac{d(2.718)}{dx} = 0.$

• If $y = 3 \times (150)^{3/5}$, then $\frac{dy}{dx} = \frac{d[3 \times 150^{3/5}]}{dx} = 0.$

2. If $y = x^n$

$$\frac{dy}{dx} = \frac{d(x^n)}{d} = n x^{n-1}$$

eg: • If $y = x^2$, then $\frac{dy}{dx} = \frac{d(x^2)}{dx}$

$$= 2x^{2-1} = 2x$$

• If $y = x^4$, then $\frac{dy}{dx} = \frac{d(x^4)}{dx}$

$$= 4x^{4-1} = 4x^3$$

• If $y = x^{3/2}$, then $\frac{dy}{dx} = \frac{d(x^{3/2})}{dx}$

$$= \frac{3}{2} x^{\frac{3}{2}-1} = \frac{3}{2} x^{1/2} = \frac{3}{2} \sqrt{x}$$

* • If $y = \sqrt{x}$, then $\frac{dy}{dx} = \frac{d(\sqrt{x})}{dx}$

$$= \frac{d(x^{1/2})}{dx} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2} \times \frac{1}{x^{1/2}} = \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

• If $y = \frac{1}{x^4}$, then $\frac{dy}{dx} = \frac{d\left(\frac{1}{x^4}\right)}{}$

$$\frac{d(x^{-4})}{dx} = -4x^{-4-1} = -4x^{-5}$$

$$= \frac{-4}{x^5}$$

- If $y = \frac{1}{x^3}$, then find instantaneous rate of change of y with respect of x. at $x = 2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d(x^{-3})}{dx}$$

$$= -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$$

when $x = 2$

$$\frac{-3}{2^4} = \frac{-3}{16}$$

3. If $y = kx^n$ where k is constant

$$\frac{dy}{dx} = \frac{d(Kx^n)}{dx}$$

$$= K\frac{d(x^n)}{dx} = K[nx^{n-1}]$$

eg: • If $y = 3x^9$, then $\frac{dy}{dx} = \frac{d(3x^9)}{dx}$

$$= 3 \times [9x^{9-1}] = 3 \times 9x^8 = 27x^8$$

- If $y = \frac{2}{3}x^{3/2}$, then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right)$

$$= \frac{2}{3} \times \left[\frac{3}{2}x^{\frac{3}{2}-1}\right] = \frac{2}{3} \times \frac{3}{2}x^{\frac{1}{2}}$$

$$= x^{1/2} = \sqrt{x}$$

- If $y = \frac{5}{3} x^{-\frac{5}{3}}$, then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{5}{3} x^{-\frac{5}{3}}\right)$

$$= \frac{5}{3}\left[\frac{-5}{3} x^{-\frac{5}{3}-1}\right] = \frac{5}{3} \times \left(\frac{-5}{3}\right) x^{-8/3}$$

$$= \frac{-25}{9} x^{-8/3} = \frac{-25}{9x^{8/3}}$$

- If $y = x$, then $\frac{dy}{dx} = \frac{d(x)}{dx}$

$$= 1x^{1-1} = 1x^{0}$$

$$= 1 \times 1 = 1$$

* $\frac{d(t)}{dt} = 1.$

- If $y = 100t$, then $\frac{dy}{dt} = \frac{d(100t)}{dt}$

$$= 100\left(\frac{dt}{dt}\right) \Rightarrow 100 \times 1 = 100.$$

*. $\frac{d(100t)}{dt} = 100$

Q-i If $y = \frac{1}{x^3}$ then find instantaneous rate of change of y with respect to x at $x = 2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d(x^{-3})}{dx}$$

$$= -3 \times x^{-3-1}$$

$$= -3 \times x^{-4}$$

$$= \frac{-3}{x^4}$$

when $x = 2$, $\frac{-3}{(-2)^4} = \frac{-3}{16}$

4. If $y = \sin x$

$$\frac{dy}{dx} = \frac{d(\sin x)}{dx} = \cos x$$

eg: • If $y = 5 \sin t$, then $\frac{dy}{dx} = \frac{d(5 \sin t)}{dt}$

$$= 5 \frac{d}{dt}(\sin t)$$

$$= 5 \cos t$$

• ~~If~~ $\cancel{y = \cos x}$, ~~then~~ $\cancel{\frac{dy}{dx} = \frac{d(\cos x)}{dx}}$

5. If $y = \cos x$

$$\frac{dy}{dx} = \frac{d(\cos x)}{dx} = -\sin(x)$$

eg: • If $y = 3 \cos(x)$, then $\frac{dy}{dx} = \frac{d(3 \cos x)}{dx}$

$$= 3(-\sin x)$$

$$= -3 \sin x.$$

• If $y = \sin 30^\circ$, then $\frac{dy}{dx} = \frac{d(\sin 30^\circ)}{dx}$.

$$= \cancel{(\cos 30^\circ = \frac{\sqrt{3}}{2}}$$

$$= \frac{d(1/2)}{dx} = 0$$

- If $y = \sin 11^\circ$, then $\frac{dy}{dx} = \frac{d(\sin 11^\circ)}{dx} = 0$.

6. If $y = \tan(x)$
$$\frac{dy}{dx} = \frac{d}{x}(\tan x) = \sec^2 x$$

7. If $y = \cot(x)$.
$$\frac{dy}{dx} = \frac{d}{x}(\cot x) = -\operatorname{cosec}^2 x$$

8. If $y = \sec(x)$
$$\frac{dy}{dx} = \frac{d}{x}(\sec x) = \sec x . \tan x$$

9. If $y = \operatorname{cosec}(x)$
$$\frac{dy}{dx} = \frac{d}{x}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x$$

10. If $y = e^x$
$$\frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x$$

eg: • If $y = 5(e^t)$, then $\frac{dy}{dt} = \frac{d(5e^t)}{dt}$
$$5\frac{d(e^t)}{dt} = 5e^t$$

11. If $y = \log_e x$
$$\frac{dy}{dx} = \frac{d}{dt}(\log_e x) = \frac{1}{x} \text{ or } x^{-1}$$

eg: • If $y = 4 \log_e t$, then $\frac{dy}{dt} = \frac{d(4\log_e t)}{dt}$

$$= 4\frac{d(\log_e t)}{dt} = \frac{4 \times 1}{t} = \frac{4}{t} = 4t^{-1}$$

12. If $y = u \pm v$ where u and v are functions of x.

$$\frac{dy}{dx} = \frac{d(u \pm v)}{dx}$$

$$= \frac{du}{dx} \pm \frac{dv}{dx}$$

+4

eg: • If $y = x^2 + \sin x$, then $\frac{dy}{dx} = \frac{d(x^2 + \sin x)}{dx}$

$$= \frac{d(x^2)}{dx} + \frac{d(\sin x)}{dx}$$

$$= 2x + \cos x$$

• If $y = (10x^3 - 20x^2 - \cos x + e^x + 5000)$, then $\frac{dy}{dx} = \frac{d(10x^3 - 20x^2 - \cos x + e^x + 5000)}{dx}$

$$= \frac{d(10x^3)}{dx} - \frac{d(20x^2)}{dx} - \frac{d(\cos x)}{dx} + \frac{d(e^x)}{dx} + \frac{d(5000)}{dx}$$

$$= 3 \times 10x^2 - 2 \times 20x + \sin x + e^x + 0$$

$$= 30x^2 - 40x + \sin x + e^x$$

Q-ii If momentum of a particle, $p = (10t^3 - 5t^2 + 20t + 50)\ \text{Kg}\,\text{ms}^{-1}$, then find F.

Force, $F = \frac{dp\ (\text{momentum})}{dt\ (\text{time})}$

+4

$$\frac{dp}{dt} = \frac{d(10t^3 - 5t^2 + 20t + 50)}{dt}$$

$$= \frac{d(10t^3)}{dt} - \frac{d(5t^2)}{dt} + \frac{d(20t)}{dt} + \frac{d(50)}{dt}$$

$$F = (30t^2 - 10t + 20)\ N$$

Q-iii A particle is moving in such a way, its displacement is represented by ~~x~~ $s = (10t^3 - 5t^2 + 20t - 50)$ m. then find.

a) velocity.

b) Acceleration.

c) velocity at $t = 1$ sec

d) Acceleration at $t = 1$ sec

e) Initial velocity

f) Initial Acceleration

a) Velocity, $v = \frac{ds\ (\text{displacement})}{dt\ (\text{time})}$

$$\frac{ds}{dt} = \frac{d(10t^3 - 5t^2 + 20t - 50)}{dt}$$

$$= \frac{d(10t^3)}{dt} - \frac{d(5t^2)}{dt} + \frac{d(20t)}{dt} - \frac{d(50)}{dt}$$

$$v = (30t^2 - 10t + 20)\ m/s.$$

b) Acceleration, $a = \frac{dv\ (\text{velocity})}{dt\ (\text{time})}$

$$\frac{dv}{dt} = \frac{d(30t^2 - 10t + 20)}{dt}$$

$$= \frac{d(30t^2)}{dt} - \frac{d(10t)}{dt} + \frac{d(20)}{dt}$$

$$A = (60t - 10)\ m/s^2$$

c) Velocity at $t = 1\,sec$

$$30t^2 - 10t + 20 = 30 - 10 + 20 = 40\ m/s.$$

d). Acceleration at $t = 1\,sec$

$$60t - 10 = 60 - 10 = 50\ m/s^2$$

e) Initial velocity, when $t = 0$

$$30t^2 - 10t + 20 = 20\ m/s.$$

f) Initial acceleration, when $t = 0$.

$$60t - 10 = -10\ m/s^2$$

Q-iv If the velocity of a particle is represented by $v = (5t^3 - 10t^2 + 20t + 5)\ m/s$. then find.

a) initial acceleration.

b) acceleration at $t = 2\ sec$

a) Acceleration, $a = \dfrac{dv\ (\text{velocity})}{dt\ .\ (\text{time})}$

$$\frac{dv}{dt} = \frac{d(5t^3 - 10t^2 + 20t + 5)}{dt}$$

$$= \frac{d(5t^3)}{dt} - \frac{d(10t^2)}{dt} + \frac{d(20t)}{dt} + \frac{d(5)}{dt}$$

$$= (15t^2 - 20t + 20)\ m/s^2$$

Initial acceleration, when $t = 0$.

$$= 20\ m/s^2$$

b). Acceleration at $t = 2sec$

$t = 2.$

$$= 15 \times 2^2 - 20 \times 2 + 20.$$

$$= 15 \times 4 - 40 + 20 = (60 - 40 + 20)\ m/s^2$$

$$= 40\ m/s^2$$

13. If $y = uv$

$$\frac{dy}{dx} = u\left(\frac{dv}{dx}\right) + v\left(\frac{du}{dx}\right)$$

eg: • If $y = x^2 \sin x$, then $\frac{dy}{dx} = \frac{d(x^2 \sin x)}{dx}$.

$$= x^2 \times \frac{d(\sin x)}{dx} + \sin x \times \frac{d(x^2)}{dx}$$

$$= x^2 \cos x + \sin x(2x)$$

$$x[x \cos x + 2 \sin x]$$

14 If $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

eg: • If $y = \frac{x}{x+1}$, then $\frac{dy}{dx} = \frac{x+1 \times \frac{d(x)}{dx} - x \times \frac{d(x+}{dx}}{(x+1)^2}$

$$= \frac{x+1 - x}{x^2 + 2x + 1} = \frac{1}{x^2 + 2x + 1}$$

• If $y = \frac{\sin x}{\cos x}$, then $\frac{dy}{dx} =$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \times \frac{d(\sin x)}{dx} - \sin x \times \frac{d(\cos x)}{dx}}{(\cos x)^2}$$

$$= \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= 1 + \tan^2 x = \sec^2 x$$

eg: • If $y = x^3 e^x$, then $\frac{dy}{dx} = \frac{d(x^3 e^x)}{dx}$

$$= x^3 \times \frac{d(e^x)}{dx} + e^x \times \frac{d(x^3)}{dx}$$

$$= x^3 \times e^x + e^x \times 3x^2$$

$$= e^x (x^3 + 3x^2)$$

$$= e^x x^2 (x+3)$$

• If $y = x^2 \cos x$, then $\frac{dy}{dx} = \frac{d(x^2 \cos x)}{dx}$

$$= x^2 \times \frac{d(\cos x)}{dx} + \cos x \times \frac{dx^2}{dx}$$

$$= x^2 \times (-\sin x) + \cos x \times 2x$$

$$= -x^2 \sin x + 2x \cos x$$

Double Differentiation:

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ (read as): a) Double differentiation of y with respect to x.

b) second derivative of y with respect to x

eg: • If $y = 4x^2 - 6x$, then find $\frac{d^2y}{dx^2}$ and $\frac{d^2y}{dx^2}$ at $x = 1$

$$\frac{d^2(4x^2 - 6x)}{dx^2} = \frac{d(4x^2)}{dx} - \frac{d(6x)}{dx}$$

$$= 8x - 6$$

$$\frac{d(8x-6)}{dx} = \frac{d(8x)}{dx} - \frac{d(6)}{dx}$$

$$= 8$$

Q-V If $y = 5e^x$, then find $\frac{d^2y}{dx^2} = ?$

$$\frac{d^2y}{dx^2} = \frac{d^2(5e^x)}{dx^2} = \frac{d(5e^x)}{dx}$$
$$= 5e^x$$

Q-vi If $x = \sin t$, then $\frac{d^2x}{dt^2} = ?$

$$\frac{d^2x}{dt^2} = \frac{d^2(\sin t)}{dt^2} = \frac{d(\cos t)}{dt}$$
$$= \cancel{-\cos t}\ -\sin t$$

Double differentiation use in Physics:

We know that,

velocity $(v) = \frac{ds}{dt}$

Acceleration $(a) = \frac{dv}{dt}$

$$a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} \text{ (Read as):}$$

~~i) Double displacement of~~
i) Double differentiation of displacement with respect to time
ii) Second derivation of displacement with respect to time

Q-vii The displacement of a particle is given by $s = 10t^3 - 20t + 50$

- Initial acceleration
- Acceleration at $t = 2$ sec

- Acceleration = $\frac{d^2s}{dt^2}$

$= \frac{d}{dt}\left(10t^3 - 20t + 50\right)$

$= \frac{d(10t^3)}{dt} - \frac{d(20t)}{dt} + \frac{d(50)}{dt}$

$= 30t^2 - 20$

$= \frac{d}{dt}(30t^2 - 20)$

$= \frac{d(30t^2)}{dt} - \frac{d(20)}{dt}$

$= 60t$

Initial acceleration = $60t = 60 \times 0 = 0$

- Acceleration at $t = 2$

$= 60 \times 2 = 120\ m/s^2$

* Motion of the partical : Non-uniform accelerated motion.

* If $a = 0$, ie. velocity is constant

Q-viii If displacement of the particle, $s = (At^3 - Bt^2 + Ct - D)$ where A, B and C are constant.

- Then find velocity at 1 sec and acceleration at 1 sec
- Initial velocity and initial acceleration.
- Type of motion

- $v = \frac{ds}{dt}$ (displacement / time)

$= \frac{d}{dt}(At^3 - Bt^2 + Ct - D)$

$= 3At^2 - 2Bt + C$ (velocity).

Acceleration $= \frac{dv}{dt}$

$= \frac{d(3At^2 - 2Bt + C)}{dt}$

$= 6At - 2B$

$v_{t=1}$ (velocity at 1sec) $= 3A - 2B + C$ and

$a_{t=1}$ (acceleration at 1sec) $= 6A - 2B$.

- Initial velocity ($v_{t=0}$) $= 3A \times 0 - 2B \times 0 + C$ $= C$.

Initial acceleration ($a_{t=0}$) $= 6 \times A \times 0 - 2B$. $= -2B$.

- Type of Motion : Non-uniform accelerated motion

Some important Questions of differentiation

i $\frac{d}{dx}[\sin(ax)] = \frac{d[\sin(ax)]}{dx} = \cos(ax) \times \frac{d(ax)}{dx}$

$= \cos(ax) \cdot a = a\cos(ax)$

eg: $\frac{d(\sin x^2)}{dt} = \cos x^2 \times \frac{d(x^2)}{dt}$

$= \cos x^2 \times 2x$

$= 2x\cos(x^2)$

Q. ix If $y = \sin wt$, then $\frac{dy}{dt} = ?$

$$\frac{d(\sin wt)}{dt} = \cancel{so}\ \cos(wt) \times \frac{d(wt)}{dt}$$

$$= \cos(wt)\, w$$

$$= w \cos(wt).$$

Q. x If $y = \sin(ax+b)$, then find $\frac{dy}{d\cancel{t}x}$

$$\frac{d(\sin ax+b)}{d\cancel{t}x} = \cos(ax+b) \times \frac{d(ax+b)}{d\cancel{t}x}$$

$$= \cos(ax+b)\, a$$

$$= a\cos(ax+b)$$

Q. xi If $y = \sin(t^4)$, then find $\frac{dy}{dt}$

$$\frac{d(\sin t^4)}{dt} = \cos(t^4) \times \frac{d(t^4)}{dt}$$

$$= \cos(t^4)\, 4t^3$$

$$= 4t^3 \cos(t^4)$$

Q. xii If $y = \sin\sqrt{x}$, then find $\frac{dy}{dx}$

$$\frac{d(\sin\sqrt{x})}{dt} = \cos(\sqrt{x}) \times \frac{d(\sqrt{x})}{dt}$$

$$= \cos\sqrt{x} \times \frac{1}{2} x^{-1/2}$$

$$= \cos\sqrt{x} \times \frac{1}{2x^{1/2}}$$

$$= \frac{1}{2\sqrt{x}} \cos\sqrt{x}$$

ii $\dfrac{d(\cos ax)}{dx} = -\sin(ax) \times \dfrac{d(ax)}{dx}$

$= -\sin(ax)\, a$

$= -a\, \sin(ax).$

Q- xiii If $y = \cos wt$, then find $\dfrac{dy}{dx}$

$\dfrac{d(\cos wt)}{dx} = -\sin(wt) \times \dfrac{d(wt)}{dx}$

$= -\sin(wt)\, w$

$= -w \sin(wt)$

Q-xiv If $y = 5(\cos 5x)$, then find $\dfrac{dy}{dx}$

$\dfrac{d(5\cos 5x)}{dx} = 5 \times (-\overset{\sin}{\cancel{\cos}}\, 5x) \times \dfrac{d(5x)}{dx}$

$= 5(-\overset{\sin}{\cancel{\cos}}\, 5x)\ 5$

$= -25\ \overset{\sin}{\cancel{\cos}}\,(5x).$

iii $\dfrac{d(e^{ax})}{dx} = e^{ax} \times \dfrac{d(ax)}{dx}$

$= a\, e^{ax}.$

eg: $\dfrac{d(e^{10x})}{dx} = e^{10x} \times \dfrac{d(10x)}{dx}$

$= e^{10x} \times 10$

$= 10 e^{10x}$

Q-xv If $y = 2e^{-5t}$, then find $\dfrac{dy}{dx}$

$\dfrac{d(2e^{-5t})}{dx} = \cancel{2e^{-5t} \times d(e^{-5t}}\ \ 2\, \dfrac{d(e^{-5t})}{dx}$

$$2\left[e^{-5t} \times \frac{d(-5t)}{dx}\right]$$

$$= 2(e^{-5t}) - 5$$

$$= -10(e^{-5t})$$

iv. $$\frac{d}{dx}[\log_e(ax)] = \frac{1}{ax} \times \frac{d}{dx}(ax)$$

$$= \frac{1}{ax} \times a = \frac{1}{x}$$

eg: • If $y = \log_e(ax+b)$, then find $\frac{dy}{dx}$.

$$\frac{d}{dx}[\log_e(ax+b)] = \frac{1}{ax+b} \times \frac{d(ax+b)}{dx}$$

$$= \frac{1}{ax+b} \times a = \frac{a}{ax+b}.$$

• If $y = \log_e(ax^2+b)$, then find $\frac{dy}{dx}$

$$\frac{d}{dx}[\log_e(ax^2+b)] = \frac{1}{ax^2+b} \times \frac{d(ax^2+b)}{dx}$$

$$= \frac{1}{ax^2+b} \times 2ax = \frac{2ax}{ax^2+b}$$

~~d~~ If $y = (ax^2+b)^5$, then find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d(ax^2+b)^5}{dx} = 5(ax^2+b)^4 \times \frac{d(ax^2+b)}{dx}$$

$$= 5(ax^2+b)^4 \times 2ax$$

$$= 10ax(ax^2+b)^4$$

Q-xvi If $y = (3x^3+5)^{7/2}$, then find $\frac{dy}{dx}$

$$\frac{d(3x^3+5)^{7/2}}{dx} = \frac{7}{2}(3x^3+5)^{5/2} \times \frac{d(3x^3+5)}{dx}$$

$$= \frac{7}{2}(3x^3+5)^{5/2} \times 9x^2$$

$$= \frac{63}{2}x^2(3x^3+5)^{5/2}$$

Q-xvii If $s \propto t^2$

$s = Kt^2$

$$v = \frac{ds}{dt} = \frac{d(Kt^2)}{dt} = 2Kt$$

$$a = \frac{dv}{dt} = \frac{d(2Kt)}{dt} = 2K$$

It has uniform acceleration i.e. constant

Q-xviii If $s \propto t^3$

$s = Kt^3$

$$v = \frac{d(s)}{dt} = \frac{d(Kt^3)}{dt} = 3Kt^2$$

$$a = \frac{d(v)}{dt} = \frac{d(3Kt^2)}{dt} = 6Kt$$

It has non-uniform acceleration because the equation of acc^n contains time (t) which changes non-uniformly.

OR

$s \propto t^3$

$\therefore v \propto 3t^2$ and

$a \propto 6t$

Hence, it has non-uniform acceleration.

Q- xix If $y = [\sin(\cos x)]$, find $\frac{dy}{dx}$

$$\frac{d[\sin(\cos x)]}{dx} = \cos(\cos x) \times \frac{d(\cos x)}{dx}$$

$$= \cos(\cos x) \times (-\sin x).$$

$$= -\sin x \cos(\cos x).$$

Q. xx If $y = \cos(\log_e x)$ find $\frac{dy}{dx}$

$$\frac{d[\cos(\log_e x)]}{dx} = -\sin(\log_e x) \times \frac{d(\log_e x)}{dx}$$

$$= -\sin(\log_e x) \times \frac{1}{x}$$

$$= -\frac{\sin(\log_e x)}{x}.$$

Q. xxi If $y = (\cos\sqrt{x})$, find $\frac{dy}{dx}$

$$= \frac{d(\cos\sqrt{x})}{dx} = (-\sin\sqrt{x}) \times \frac{d(\sqrt{x})}{dx}$$

$$= -\sin\sqrt{x} \times \frac{d(x^{1/2})}{dx}$$

$$= -\sin\sqrt{x} \times \frac{1}{2} x^{-1/2}$$

$$= -\sin\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}} \sin\sqrt{x}$$

Q. xxii If $y = \log_e(e^x)$, find $\frac{dy}{dx}$.

$$= \frac{1}{e^x} \times \frac{d(e^x)}{dx}$$

$$= \frac{1}{e^x} \times e^x = 1$$

To find the slope by differentiate:

- Slope (m) = $\tan\theta = \frac{dy}{dx}$ (slope of y-x graph)

Q.xxiii If $y = 2x+1$, then find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d(2x+1)}{dx}$$

$$= \frac{d(2x)}{dx} + \frac{d(1)}{dx}$$

$$= 2 + 0 = 2$$

Q.xxiv If $y = x^2$, then find slope of y-x graph at $x = 2$

$$\frac{dy}{dx} = \frac{d(x^2)}{dx} = 2x$$

when $x = 2$

$$2x = 2\times 2 = 4$$

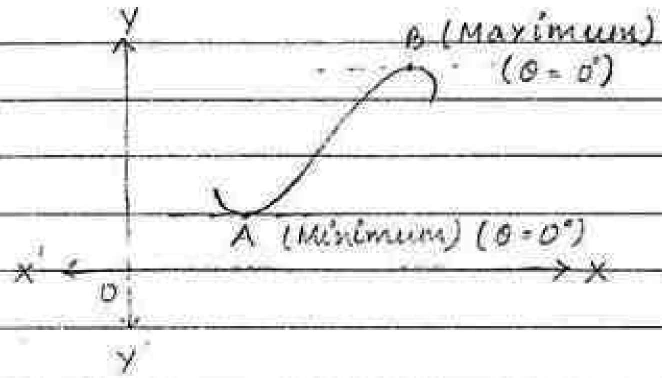
Q-xxv If $y = 10x^3 - 5x^2 + 2x - 2$, then find slope of y-x graph at $x=1$.

$$\frac{dy}{dx} = \frac{d(10x^3 - 5x^2 + 2x - 2)}{dx}$$

$$= \frac{d(10x^3)}{dx} - \frac{d(5x^2)}{dx} + \frac{d(2x)}{dx} - \frac{d(2)}{dx}$$

$$= 30x^2 - 10x + 2 \text{, at } x=1 = 30-10+2 = 22$$

Maxima and Minima:



- Conditions for minima:

Slope $(m) = \frac{dy}{dx} = \tan\theta = \tan 0°$

$\therefore \theta = 0°$

$\therefore \frac{dy}{dx} = 0$

- Conditions for maxima:

Slope $(m) = \frac{dy}{dx} = \tan\theta = \tan 0$

$\therefore \theta = 0°$

$\therefore \frac{dy}{dx} = 0$

* Conclusion:

1st derivative of y with respect to x is always 0 for minima as well as maxima

- To find minima:

i) $\frac{dy}{dx} = 0.$

ii $\frac{d^2y}{dx^2} = +ve$

• To find maxima.

i) $\frac{dy}{dx} = 0$

ii) $\frac{d^2y}{dx^2} = -ve.$

Q.xxvi If $y = 5x^2 - 2x + 1$, then find minimum value of y

$$\frac{dy}{dx} = \frac{d(5x^2 - 2x + 1)}{dx}$$

$$= 10x - 2$$

$\because$ for minima $\frac{dy}{dx} = 0.$

$$\therefore 10x - 2 = 0.$$

$$x = \frac{2}{10} = \frac{1}{5}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{5}} = \left(\frac{d(10x-2)}{dx}\right)_{x=\frac{1}{5}} = 10.$$

$\because$ 10 is (+ve).

$\therefore$ the value obtained will be minimum.

$$y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1$$

$$= 5 \times \frac{1}{25} - \frac{2}{5} + 1$$

$$= \frac{1-2+5}{5} = \frac{6-2}{5} = \frac{4}{5}$$

Q-xxvii If $y = 2x^3 - 9x^2 + 12x + 15$, then find maximum and minimum value of y

$$\frac{dy}{dx} = \frac{d(2x^3 - 9x^2 + 12x + 15)}{dx}$$

$$= 6x^2 - 18x + 12$$

for minima and maxima $\frac{dy}{dx} = 0.$

$$\therefore 6x^2 - 18x + 12 = 0.$$
$$6(x^2 - 3x + 2) = 0.$$
$$x^2 - 3x + 2 = 0.$$
$$x^2 - 2x + x + 2 = 0$$
$$x(x-2) - (x-2) = 0.$$
$$x = 2 \quad \text{or} \quad x = 1.$$

~~$\because$ 2 and 1 are (+ve)~~

~~$\therefore$ the value obtained from either of them will be minimum for y.~~

To check i)

$$\left(\frac{d^2y}{dx^2}\right)_{x=2} = \left[\frac{d(6x^2 - 18x + 12)}{dx}\right]_{x=2}$$
$$= [12x - 18]_{x=2}$$
$$= 12 \times 2 - 18 = 24 - 18 = +6$$

$\because$ 6 is (+ve).

$\therefore$ the value obtained will be minimum for y

To check ii)

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = \left[\frac{d(6x^2 - 18x + 12)}{dx}\right]_{x=1}$$
$$= [12x - 18]_{x=1}$$
$$= 12 - 18 = -6$$

$\because$ 6 is (-ve)

$\therefore$ the value obtained will be maximum for y

Hence, $\begin{bmatrix} y_{min} \\ \text{at } 2 \end{bmatrix} = 2(2)^3 - 9(2)^2 + 12x + 15$

$$= 2 \times 8 - 9 \times 4 + 24 + 15$$
$$= 1 + 24 + 15 - 36 = 19 \text{ and}$$

Y_{max} at $x = 1$.

$= 2(1)^3 - 9(1)^2 + 12(1) + 15$

$= 2 - 9 + 12 + 15$

$= 29 - 9 = 20.$

$\therefore Y_{min} = 19$ and $Y_{max} = 20.$

Qxxviii If $y = 2x - x^2 + 1$, then find the maximum value of y

$$\frac{dy}{dx} = \frac{d(2x - x^2 + 1)}{dx}$$

$= 2 - 2x$

$\because \frac{dy}{dx} = 0$ for maximum value of y

$\therefore 2 - 2x = 0.$

$x = \frac{-2}{-2} = 1.$

$$\frac{d^2y}{dx^2} = \left[d.(2-2x)\right]_{x=1}$$

$= (-2)\ x = 1.$

$\because -2$ is $(-ve)$

$\therefore (Y_{max})_{x=1}$

Hence, $y_{max} = 2(1) - (1)^2 + 1$

$= 2 - 1 + 1$

$= 2$

Qxxix Find the minimum value of the particle if $v = t^2 - 2t + 10$

$$\frac{dy}{dx} = \frac{d(t^2 - 2t + 10)}{dx}$$

$= 2t - 2$

$\because \frac{dy}{dx} = 0$ for minimum value of y.

$\therefore \quad 2t - 2 = 0$

$t = \frac{2}{2} = 1$

$\frac{d^2y}{dx^2} = \left[d(2t-2) \right]_{t=1}$

$= 2$

$\because$ the value obtained is (+ve).

$\therefore$ minimum value of y is at $x = 1$.

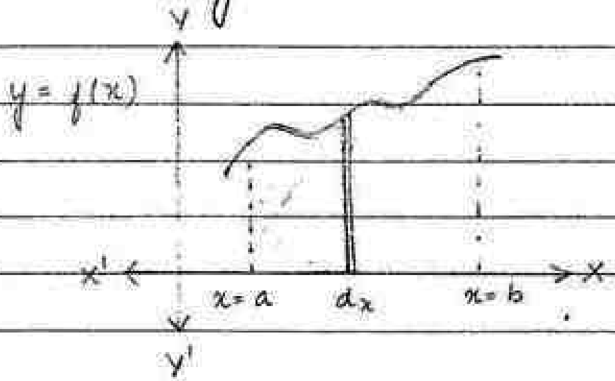
$y_{min} = (1)^2 - 2(1) + 10$

$= 1 - 2 + 10$

$= 11 - 2 = 9\ m/s.$

INTEGRATION

- It is the reverse process of differentiation.
- It is used to sum up the quantity which vary continuously.



- Area of small part or elementary part, $y = dx$
- Total area between graph (curve) x-axis $= \int y\,dx$

$\int y\,dx$: (read as) :

- Integration of y with respect to x, where y is the function of x

OR

- Area between ~~y x axis~~ graph & x-axi

- Total Area from $x = a$ to $x = b$

$\int_a^b y\,dx$: (read as) :

- Area between graph and x-axis from $x = a$ to $x = b$

Types of Integration :

i Indefinite integration :
In this type of integration, limits are not specified and we add 'c' with the final answer, where c is known as

'constant of integration' or 'Integration constant'

ii Definite Integration:
In this type of integration, limits are specified and we dont add 'c' with the final answer.

Some basic formulae of integration.

i. if $y = \frac{1}{x}$ or x^{-1}

$$\int y \, dx = \int \frac{1}{x} dx = \log_e(x) + c$$

ii If $y = \cos x$

$$\int y \, dx = \int (\cos x) \, dx = \sin x + c$$

iii If $y = \sin x$

$$\int y \, dx = \int (\sin x) \, dx = -\cos x + c$$

iv If $y = e^x$

$$\int y \, dx = \int (e^x) \, dx. = e^x + c$$

v. If $y = K$, where K is constant.

$$\int y \, dx = \int K \, dx = K \int dx. = K(x) + C = Kx + C$$

eg: $\int dt = t + c$

$\int dy = y + c$

$\int dv = v + c$

$\int dm = m + c$

$\int dp = p + c$

$\int dI = I + c$

$\int 10\,dx = 10x + c$

$\int (-100)\,dt = -100t + c$

$\omega \int dt = \omega t + c$

$\pi \int dx = \pi x + c$

* $\int 2014\,dm = 2014m + c$

vi If $y = x^n$

$$\int y\,dx = \int (x^n)\,dx = \frac{x^{n+1}}{n+1} + c \quad \left[\begin{array}{l}\text{where,} \\ x \neq -1\end{array}\right]$$

eg: $\int (x^4)\,dx = \frac{x^{4+1}}{4+1} + c = \frac{x^5}{5} + c$

$$\int (x^{3/2})\,dx = \frac{x^{3/2+1}}{3/2+1} + c = \frac{x^{5/2}}{5/2} + c$$

$$= \frac{2x^{\frac{5}{2}}}{5} + c$$

$$\int (\sqrt{x})\,dx = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c$$

$$= \frac{2x^{\frac{3}{2}}}{3} + c$$

$$\int (x^{-4})\,dx = \frac{x^{-4+1}}{-4+1} + c = \frac{x^{-3}}{-3} + c$$

$$= -\frac{1}{3x^3} + c$$

* $\int \left(\frac{1}{x^2}\right) dx = \int (x^{-2})\,dx = \frac{x^{-2+1}}{-2+1} + c$

+4

$$= \frac{x^{-1}}{-1} + c = \frac{-1}{x} + c$$

Similarly $\int \left(\frac{1}{r^2}\right) dx = \frac{-1}{r} + c$

vii. If $y = kx^n$, where k is a constant.

$$\int y\,dx = \int (kx^n)\,dx = k\int x^n\,dx. \quad \left[\begin{array}{l}\text{where,}\\ n \neq -1\end{array}\right]$$

$$= \frac{kx^{n+1}}{n+1} + c$$

eg: $\int 5x^5\,dx = \frac{5x^{5+1}}{5+1} + c = \frac{5x^6}{6} + c$

$$\int 100x^{-3/2}\,dx = \frac{100x^{-3/2+1}}{-3/2+1} + c$$

$$= \frac{100x^{-1/2}}{-1/2} + c = \frac{-200}{\sqrt{x}} + c$$

Some important questions of Integration:

i. If $y = \sin(ax)$

$$\int [\sin(ax)]\,dx = -\frac{\cos ax}{a} + c \quad \left[\begin{array}{l}\text{where}\\ x^1 = x\end{array}\right]$$

eg: $\int \sin(\omega t)\,dt = -\frac{\cos \omega t}{\omega} + c$

$$\int 10\sin(100x)\,dx = \frac{10[-\cos(100x)]}{100} + c$$

$$= -\frac{\cos(100x)}{10} + c$$

$$\int 5\sin(10x-5)\,dx = \frac{5[-\cos(10x-5)]}{10} + c$$

$$= -\frac{\cos(10x-5)}{2} + c$$

$$\int \omega\sin(\omega t-\phi)\,dt = \frac{\omega[-\cos(\omega t-\phi)]}{\omega} + c$$

$$= -\cos(\omega t-\phi) + c$$

ii If $y = \cos(ax)$,

$$\int [\cos(ax)]\,dx = \frac{\sin(ax)}{a} + c$$

eg:

$$\int \cos(\omega t)\,dx = \frac{\sin(\omega t)}{\omega} + c$$

$$\int 100\cos(100x-100)\,dx = \frac{100\sin(100x-100)}{100} + c$$

$$= \sin(100x-100) + c$$

$$5\int \cos(7t-5)\,dt = \frac{5\sin(7t-5)}{7} + c$$

iii If $y = e^{ax}$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$$

eg: $$\int e^{10x}\,dx = \frac{e^{10x}}{10} + c$$

$$\int -10e^{-100t}\,dt = \frac{-10\,e^{-100t}}{-100} + c$$

$$= \frac{e^{-100t}}{10} + c$$

$$\int e^{5}\,dt = e^{5}t + c$$

iv If $y = (ax+b)^n$ [where $n \neq -1$]

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

eg: $$\int (2x-9)^{7/2}\,dx = \frac{(2x-9)^{9/2}}{9/2 \times 2} + c$$

$$= \frac{(2x-9)^{9/2}}{9} + c$$

v If $y = (u \pm v)$, where u and v are function of x.

$$\int (u \pm v)\, dx = \int u\, dx \pm \int v\, dx + c$$

eg: $\int [5x^3 - \sin(2x)]\, dx = \frac{5x^{3+1}}{3+1} - \left[\frac{-\cos(2x)}{2}\right] + c$

$$= \frac{5x^4}{4} + \frac{\cos(2x)}{2} + c.$$

$$\int \left(t^3 - \cos^t + e^t + \frac{1}{t}\right) dx =$$

$$= \frac{t^{3+1}}{3+1} - \frac{\sin^{t+1}}{t+1} + e^t + t^{-1} + c$$

$$= \frac{t^4}{4} - \left[\frac{\sin^{t+1}}{t+1}\right] + e^t + \log_e t + c$$

Use of integration in Physics:

i velocity, $v = \frac{d(s)}{dt}$

$\therefore d(s) = v.dt$

Integrate both side

$\int ds = \int v dt$

$\therefore s = \int v dt$

ii acceleration, $a = \frac{d(v)}{dt}$

$\therefore d(v) = a.dt$

Integrate both sides.

$\int dv = \int a dt$

$\therefore v = \int a dt$

$$\begin{array}{ccccc} & \xrightarrow{\text{differentiation}} & & \xrightarrow{\text{differentiation}} & \\ \text{Displacement} & & \text{Velocity} & & \text{Acceleration.} \\ (s) & & (v) & & (a) \\ & \xleftarrow[\text{integration}]{} & & \xleftarrow[\text{integration}]{} & \end{array}$$

Q-1 If acceleration of a particle, $a = t^4 - 3t^2 + 2t + 5$ m/s^2, then find velocity of the particle.

velocity, $v = \int a\,dt$

$= \int (t^4 - 3t^2 + 2t + 5)\,dt$

$v = \left(\frac{t^5}{5} - \frac{3t^3}{3} + \frac{2t^2}{2} + 5t + c \right)$ m/s.

$= \left(\frac{t^5}{5} - t^3 + t^2 + 5t + c \right)$ m/s.

Q.2 If the velocity of the particle is $v = 4t^2 - 5t + 100$ m/s, then find:

i) displacement

ii) acceleration.

i) displacement, $s = \int v\,dt$

$= \int (4t^2 - 5t + 100)\,dt$

$= \left(\frac{4t^3}{3} - \frac{5t^2}{2} + 100t + c \right)$ m.

ii acceleration, $a = \frac{dv}{dt}$.

$= \frac{d(4t^2 - 5t + 100)}{dx}$

$= (8t - 5)\ m/s^2$

Definite Integration :

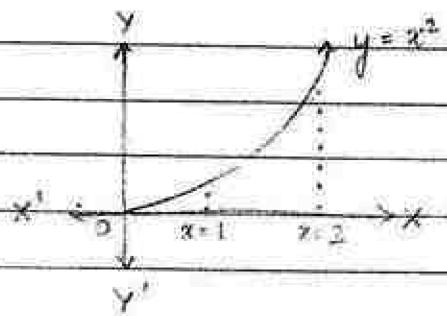
$$\int_a^b y\,dx = \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$

$$= F(b) - F(a)$$

eg: $\displaystyle\int_1^2 x^2 \cdot dx = \left[\frac{x^3}{3}\right]_1^2$

$$= \frac{(2)^3}{3} - \frac{(1)^3}{3}$$

$$= \frac{8-1}{3} = \frac{7}{3} \text{ units}$$



$$\int_2^1 x^2\,dx = \left[\frac{x^3}{3}\right]_2^1$$

$$= \cancel{(2)}\ \frac{(1)^3}{3} - \frac{(2)^3}{3}$$

$$= \frac{1-8}{3} = \frac{-7}{3} \text{ units}$$

* If we interchange the limit of the integration, the sign of the answer also gets reversed.

Q.3 $\displaystyle\int_2^3 x^3\,dx = \left[\frac{x^4}{4}\right]_2^3$

$$= \frac{(3)^4}{4} - \frac{(2)^4}{4}$$

$$= \frac{81-16}{4} = \frac{65}{4}$$

Q.4 $\int_{-1}^{1} t^3 dt = \left[\frac{t^4}{4}\right]_{-1}^{1}$

$$= \frac{(1)^4}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = 0.$$

Q.5. $\int_{1}^{2} (t+t^2)\, dt = \int_{1}^{2} (t+t^2) dt.$

$$= \left[\frac{t^2}{2} + \frac{t^3}{3}\right]_1^2 = \left[\frac{2^2}{2} + \frac{2^3}{3}\right] - \left[\frac{1^2}{2} + \frac{1^3}{3}\right]$$

$$= \left(\frac{4\times3 + 8\times2}{6}\right) - \left(\frac{3+2}{6}\right) = \frac{28-5}{6} = \frac{23}{6}$$

Q.6 $\int_{0}^{\pi/4} \sin x \, dx. = \left[-\cos x\right]_0^{\pi/4}$

$$= \left[-\cos x\right]_{0.}^{180/4}$$

$$= (-\cos 45^\circ) - (-\cos 0^\circ)$$

$$= -\frac{1}{\sqrt{2}} - (-1)$$

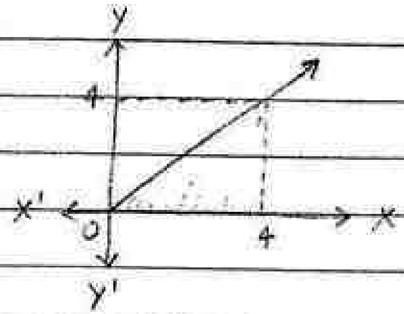
$$= 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}$$

Q.7 $\int_{-\pi/2}^{\pi/2} \cos x \, dx = \left[\sin x\right]_{-180/2}^{180/2}$

$$= \left[\sin x\right]_{-90}^{90^\circ}$$

$$= \sin(90^\circ) - \sin(90^\circ) = 1+1 = 2$$

Q-8



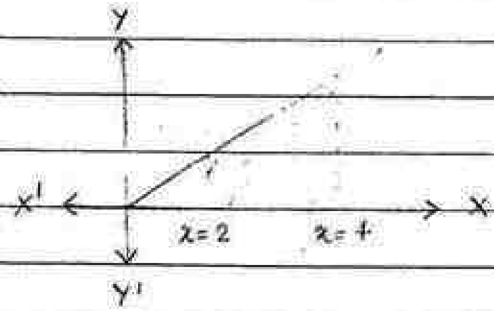
Find the area of shaded region of $y = x$.

$$= \int_0^4 x \cdot dx = \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{(4)^2}{2} - \frac{(0)^2}{2}$$

$$= \frac{16}{2} - 0 = 8 \text{ sq units.}$$

Q.9. If $y = 2x$, then find area between graph and $x = 2$ to $x = 4$



$$\int_2^4 2x \cdot dx = \left[\frac{2x^2}{2}\right]_2^4$$

$$= (4)^2 - (2)^2$$

$$= 16 - 4$$

$$= 12 \text{ units.}$$

Q.10 The acceleration of a particle is $a = 2(t-1)$ The velocity of the particle at $t = 5$ and the initial velocity $= t = 10$ m/s.

$\because a = 2(t-1) = 2t - 2$

$$\frac{dv}{dt} = 2t - 2$$

$$\int_{10}^{v} dv = \int_0^5 (2t-2)\,dt$$

$$[v]_{10}^{v} = \left[\frac{2t^2}{2} - 2t\right]_0^5$$

$$= (v - 10) = \left(\frac{2\times5^2}{2} - 2\times5\right) - 0$$

$$= v - 10 = 25 - 10$$

$\therefore$ $v = 25$ m/s.

Trick :
When initial velocity is given and we are supposed to find final velocity after time t, then,
integrate ~~integrate~~ the equation of acceleration and add initial velocity to it

Q.11 The initial velocity of an particle is u at $t = 0$ and the acceleration is given by $f = at^3$. then find the final velocity.

$$f = at^3$$

$$\frac{dv}{dt} = at^3$$

$$\int_u^v dv = \int_0^t at^3\,dt$$

$$[v]_u^v = \left[\frac{at^4}{4}\right]_0^t$$

$$v - u = \frac{at^4}{4} - 0$$

$$\therefore v = \frac{at^4}{4} + u$$

Using trick,

$$v = \int at^3\,dt + u$$
$$= \frac{at^4}{4} + u.$$

- If $a = \frac{dv}{dt}$

$$a = \frac{dv}{dt} \times \frac{ds}{ds}$$
$$= \frac{dv}{ds} \times \frac{ds}{dt}$$
$$= \frac{dv}{ds} \times v = v\left(\frac{dv}{ds}\right)$$

This formulae can be used to find acceleration when velocity & displacement relation is given.

eg: If $v = t^2$, find a

$$a = \frac{dv}{dt} = \frac{d(t^2)}{t} = 2t$$

If $v = s^2$, find a

$$a = v.\frac{dv}{ds} = s^2 \times \frac{d(s^2)}{ds}$$
$$= s^2 \times 2s = 2s^3.$$