PROJECTILE MOTION

- When a body is projected such that velocity of projection is not collinear to the force (i.e. acceleration), then body moves along curved path. But if the force (i.e. acceleration) on the body is constant, then curved path of the body is parabolic.

- The horizontal distance travelled by the projectile during its motion is called ~~p~~ horizontal range.

- Projectile motion is an example of two dimensional motion with constant acceleration. ($a_{net} = g$)

☆ • Projectile motion is the combination of two mutally perpendicular motions which are horizontal motion and vertical motion. Both these motions are independent from each other because component of any vector along its perpendicular direction is 0.

- In the horizontal direction, there is no force on the projectile. Due to its horizontal acceleration is zero, means the horizontal component of velocity remains constant throughout the projectile motion.

$\because F_x = 0.$

$\therefore a_x = \frac{F_x}{m} = \frac{0}{m} = 0$

$\therefore \frac{d(v_x)}{dt} = 0$

$\therefore v_x = \text{constant}$

- At the topmost position, vertical velocity is 0, but horizontal velocity is non zero.

  $\therefore (\text{speed})_{Top} = |\vec{v}_{top}| = \sqrt{v_x^2 + v_y^2}$

  $= \sqrt{(u\cos\theta)^2 + 0^2} = u\cos\theta$

- The angle between $\vec{v}$ and $\vec{a}$ is 90° at the topmost position.

- Horizontal component of velocity, acceleration, force and total mechanical energy (PE + KE) remains constant

- Vertical component of velocity, speed, velocity, momentum, KE and PE do not remain constant.

- KE, speed, velocity at topmost position and momentum becomes minimum but P.E becomes maximum.

Q-1 Projectile Motion supplement (PMS) Pg-2

i) Change in speed $= v_f - v_i$

$= u - u = 0$

ii) Change in velocity $= \Delta\vec{v} = \vec{v_f} - \vec{v_i}$

$$\left[\begin{array}{l} \because \vec{v_i} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} \\ \vec{v_f} = u\cos\theta\,\hat{i} - u\sin\theta\,\hat{j} \end{array}\right]$$

$$\begin{aligned} \therefore \ \Delta\vec{v} &= \vec{v_f} - \vec{v_i} \\ &= (u\cos\theta\,\hat{i} - u\sin\theta\,\hat{j}) - (u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}) \\ &= u\cos\theta\,\hat{i} - u\sin\theta\,\hat{j} - u\cos\theta\,\hat{i} - u\sin\theta\,\hat{j} \\ &= -2u\sin\theta\,\hat{j} \\ &= 2u\sin\theta \text{, vertically downwards.} \end{aligned}$$

iii) Change in momentum $= \Delta\vec{p} = \Delta(m\vec{v}) = m(\Delta\vec{v})$

$$\begin{aligned} &= m(-2u\sin\theta\,\hat{j}) \\ &= -2mu\sin\theta\,\hat{j} \\ &= 2mu\sin\theta \text{, vertically downwards.} \end{aligned}$$

Q.2 (PMS) Pg-2

i) change in speed = Δ speed = $v_f - v_i$

$$= u\cos\theta - u$$

ii) change in velocity = $\Delta\vec{v} = \vec{v_f} - \vec{v_i}$

$$\left[\begin{array}{l} \vec{v_i} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} \\ \vec{v_f} = u\cos\theta\,\hat{i} \end{array}\right]$$

$$\begin{aligned} \because \ \Delta\vec{v} &= \vec{v_f} - \vec{v_i} \\ &= (u\cos\theta\,\hat{i}) - (u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}) \\ &= u\cos\theta\,\hat{i} - u\cos\theta\,\hat{i} - u\sin\theta\,\hat{j} \\ &= -u\sin\theta\,\hat{j} \\ &= u\sin\theta \text{, vertically downward} \end{aligned}$$

iii) change in momentum $= \Delta\vec{p} = \Delta(m\vec{v}) = m(\Delta\vec{v})$

$$\begin{aligned} &= m(-u\sin\theta\,\hat{j}) = -mu\sin\theta\,\hat{j} \\ &= mu\sin\theta \text{, vertically downward} \end{aligned}$$

Q.3. (PMS) Pg - 2

Let mass = m and initial speed = u

$(KE)_{ground} = \frac{1}{2} mv^2 = E$

$\therefore \; E = \frac{1}{2} mu^2$

$(KE)_{top} = \frac{1}{2} m V_{top}^2 = \frac{1}{2} m (u \cos \theta)^2$

$= \frac{1}{2} mu^2 \cos^2 \theta$

$= E \cos^2 \theta$

* $\boxed{(KE)_{top} = (KE)_{ground} \times \cos^2 \theta}$

Time of flight (T) :

* $T = \frac{2u_y}{g} = \frac{2u \sin \theta}{g}$

Horizontal Range (R) :

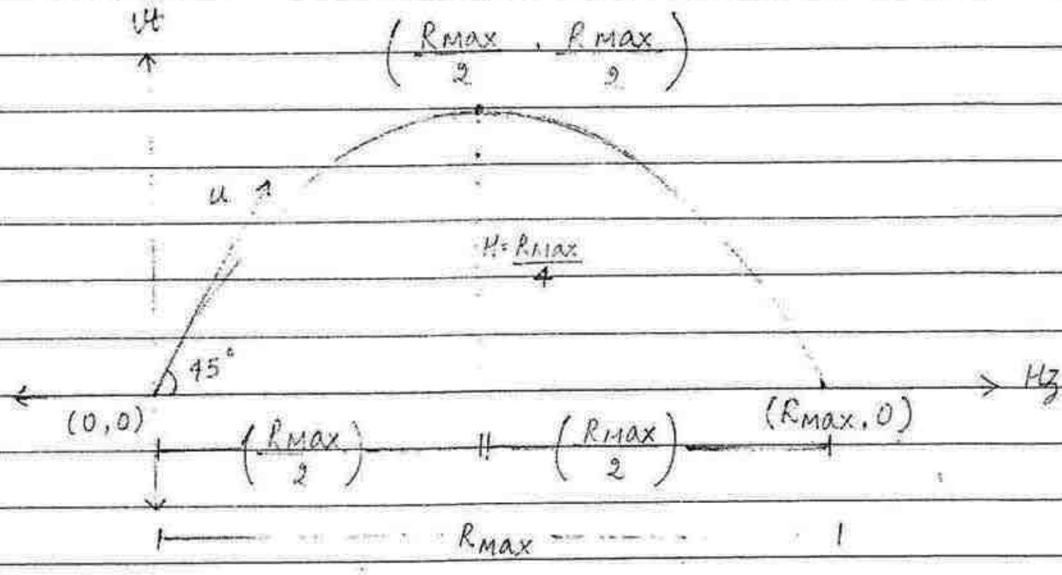
* $R = \frac{2u_x u_y}{g} = \frac{u^2 \sin 2\theta}{g} = u_x T$

* $R_{max} = \frac{u^2}{g}$ (At $\theta = 45^\circ$)

Maximum height attained (H) :

* $H = \frac{u_y^2}{2g} = \frac{u^2 \sin^2 \theta}{2g} = \frac{T^2 g}{8}$

$*$ $H = \frac{R_{max}}{4}$ (At $\theta = 45^\circ$)



$*$ $H_{max} = \frac{u^2}{2g}$ (At $\theta = 90^\circ$) $= \frac{R_{max}}{2}$ (at $\theta = 45^\circ$)

~~Velocity-time~~ Velocity - triangle after 't' time

- Velocity of 't' time $= \vec{v} = v_x\hat{i} + v_y\hat{j}$
  $= u_x\hat{i} + (u_y - gt)\hat{j}$
  $u\cos\theta\,\hat{i} + (u\sin\theta - gt)\hat{j}$

- Speed after 't' time $= \sqrt{v_x^2 + v_y^2}$
  $= \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$

- Angle of velocity from Horizontal after 't' time is given by $= \tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$

Q.4 (PMS) Pg - 4

$(\text{Speed})_{top} = \frac{1}{\sqrt{2}} u$

$u \cos \theta = \frac{1}{\sqrt{2}} u$

$\therefore \cos \theta = \frac{1}{\sqrt{2}} = \cos 45^\circ$

$\therefore \theta = 45^\circ$

$R_{Max} = \frac{u^2 \sin 2\theta}{g}$

$= \frac{u^2 \times 1}{g} = \frac{u^2}{g}$

Q.5 (PMS) Pg - 4

$R_1 = 1000 \text{ m}, \theta = 75^\circ$

$R_1 = \frac{u^2 \sin 2\theta}{g}$

$1000 = \frac{u^2}{g} \sin 75 \times 2$

$\therefore \frac{u^2}{g} = \frac{1000}{\sin 150}$

$= \frac{1000 \times 2}{1}$ $\left[\because \sin 150^\circ = \frac{1}{2}\right]$

$= 2000$

$R_2 = \frac{u^2}{g} \sin 2\theta$

$= 2000 \times \sin 2 \times 30^\circ$ $\left[\because \frac{u^2}{g} = 2000\right]$

$= 2000 \times \sin 60^\circ$

$$2000 \times \frac{\sqrt{3}}{2} = 1000 \times \sqrt{3}$$

$$= 1000 \times 1.732$$

$$= 1732 \text{ m}$$

Q.6 $H_1 = 300 \text{ m}, \quad \theta = 60^\circ$

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

$$300 = \frac{u^2}{2g} \sin^2 60^\circ$$

$$\frac{u^2}{2g} = \frac{300 \times 4}{3}$$

$$= 100 \times 4$$

$$= 400$$

$$H_2 = \frac{u^2}{2g} \sin^2 \theta$$

$$= 400 \times \sin^2 30^\circ$$

$$= 400 \times \frac{1}{4} = 100 \text{ m}$$

Q.7 (PMS) Pg - 4

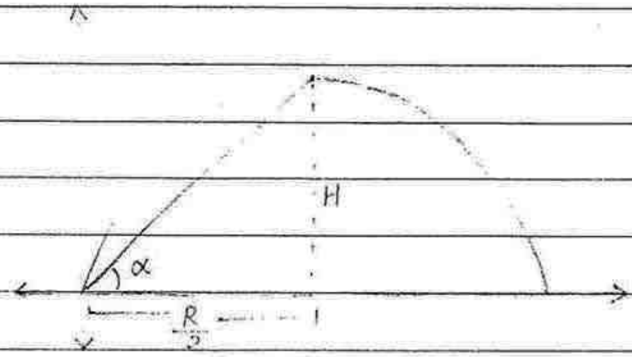
$$\frac{R}{H} = \frac{u^2 \sin 2\theta}{g} \times \frac{2g}{u^2 \sin^2 \theta}$$

$$= \frac{2 \times \sin 2 \times 45^\circ}{\sin^2 45^\circ}$$

$$= \frac{2 \times \sin 90^\circ}{\sin^2 45^\circ}$$

$$= \frac{2 \times 1 \times 2}{1} = \frac{4}{1}$$

$$\therefore R_{max} : H = 4 : 1$$



Angle of elevation = $\tan\alpha = \dfrac{H}{\frac{R}{2}}$

$\tan\alpha = \dfrac{2H}{R} = 2 \times \dfrac{1}{4}$ $\left[\because \dfrac{R}{H} = \dfrac{4}{1}\right]$

$\tan\alpha = \dfrac{1}{2}$

$\therefore \alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$

Q.8 (PMS) Pg - 4.

$\because H = \dfrac{R_{max}}{4}$

$H = \dfrac{500m}{4} = 125\ m$

Q.9 (PMS) Pg - 4.

$\because H_{max} = \dfrac{u^2}{2g} = \dfrac{R_{max}}{2}$ $\left[\because R_{max} = \dfrac{u^2}{g}\right]$

$\therefore \dfrac{R_{max}}{2} = \dfrac{500}{2}$

$= 250m$

Q.10 (PMS) Pg-4

$\because H_{max} = \frac{u^2}{2g} = \frac{R_{max}}{2}$ $\left[\because R_{max} = \frac{u^2}{g}\right]$

$\therefore \frac{R_{max}}{2} = \frac{1000}{2} = 500\,m$

Q.11 (PMS) Pg-4

$\because H_{max} = \frac{u^2}{2g} = \frac{R_{max}}{2} = 1000$ $\left[\because R_{max} = \frac{u^2}{g}\right]$

$\therefore R_{max} = 1000 \times 2\ m$

$= 2000\ m$

Q.12 (PMS) Pg-4

i) $\frac{T_A}{T_B} = \frac{2u \sin\theta}{g} \div \frac{2u \sin(90-\theta)}{g}$

$= \frac{2u \sin\theta}{g} \times \frac{g}{2u\cos\theta}$ $\left[\because \sin(90-\theta) = \cos\theta\right]$

$= \frac{\sin\theta}{\cos\theta} = \tan\theta$

ii) $\frac{H_A}{H_B} = \frac{u^2 \sin^2\theta}{2g} \div \frac{u^2 \sin^2(90-\theta)}{2g}$

$= \frac{u^2 \sin^2\theta}{2g} \times \frac{2g}{u^2 \cos^2\theta}$ $\left[\because \sin^2(90-\theta) = \cos^2\theta\right]$

$= \frac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$

iii) $\frac{R_A}{R_B} = \left(\frac{2u_x u_y}{g}\right)_A \div \left(\frac{2u_x u_y}{g}\right)_B$

$\because$ For A = $u_x = u\cos\theta$ and
$u_y = u\sin\theta$
but
For B = $u_x = u\sin\theta$ and $u_y = u\cos\theta$, (from vertical)

$\therefore \frac{2\times(u\cos\theta)(u\sin\theta)}{g} \times \frac{g}{2(u\sin\theta)(u\cos\theta)}$

$= 1$

$\therefore R_A = R_B$

OR

$\frac{R_A}{R_B} = \frac{u^2\sin 2\theta}{g} \div \frac{u^2\sin 2(90-\theta)}{g}$

$= \frac{u^2\sin 2\theta}{g} \times \frac{g}{u^2\sin(180-2\theta)}$

$= \frac{\sin 2\theta}{\sin(180-2\theta)}$

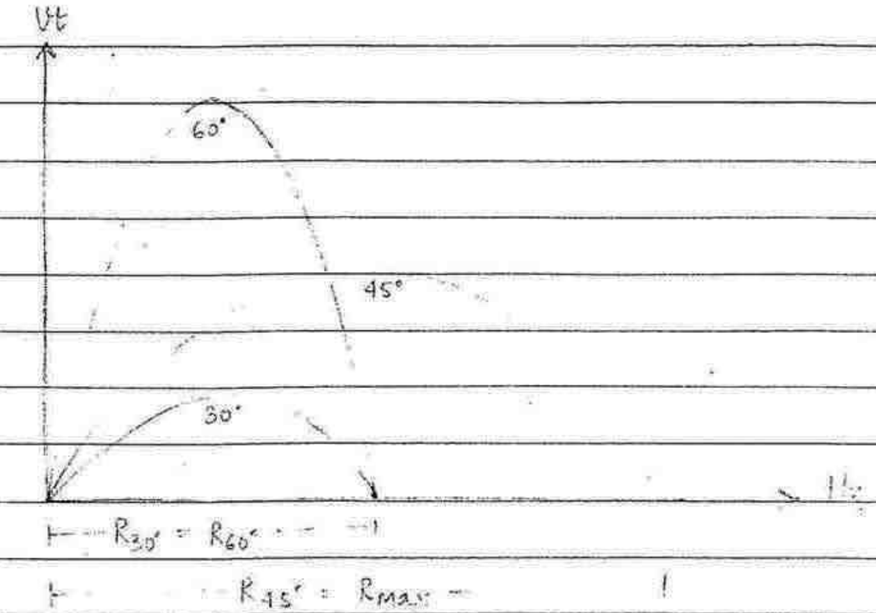
$= \frac{\sin 2\theta}{\sin 2\theta} = 1 \quad [\because \sin(180^\circ-2\theta) = \sin 2\theta]$

$\therefore R_A = R_B$.

* If two projectile are thrown with same speed at complementary angle of projection then their horizontal ranges are equal

If $u_1 = u_2$, then $R_1 = R_2$

If θ_1	30°	15°	$45^\circ + \alpha$
θ_2	$90^\circ - 30^\circ$ $= 60^\circ$	$90^\circ - 15^\circ$ $= 75^\circ$	$90^\circ - (45^\circ + \alpha)$ $= 45^\circ - \alpha$



If $R = nH$

i.e. $\dfrac{u^2\, 2\sin\theta\cos\theta}{g} = n \times \dfrac{u^2 \sin^2\theta}{2g}$

$= 2\cos\theta = n \times \dfrac{\sin\theta}{2}$

$\therefore \dfrac{2\times 2}{n} = \dfrac{\sin\theta}{\cos\theta}$

☆ $\therefore \boxed{\tan\theta = \dfrac{4}{n}}$

Equation of Trajectory:

(PMS) Pg-6

☆ $\boxed{y = x\tan\theta\left[1 - \dfrac{x}{R}\right]}$

Q.13 (PMS) Pg-7

$\because y = ax - bx^2$

$\cancel{x(a-bx)}$ $ax\left(1 - \dfrac{bx}{a}\right)$

$$\therefore\ y = ax\left[1 - \frac{bx}{a}\right] \Rightarrow ax\left[1 - \frac{x}{a/b}\right]$$

By comparing,

$$\because\ y = x\tan\theta\left[1 - \frac{x}{R}\right]$$

we get,

$$\tan\theta = a$$

$$\therefore\ \theta = \tan^{-1}(a) \text{ and.}$$

$$R = \frac{a}{b.}$$

Q.14 (PMS, Pg - ?)

i) $(KE)\,\text{ground} = \cancel{\frac{1}{m}}\ \frac{1}{2}mv^2 = \frac{1}{2}mu^2 \quad (\because v = u)$

ii) $(ME)\,\text{ground} = (KE)\,\text{ground} + (PE)\,\text{ground.}$

$$= \frac{1}{2}mu^2 + 0 \quad (\because PE\,\text{ground} = 0)$$

$$= \frac{1}{2}mu^2$$

iii) $(PE)\,\text{top} = mgH = mg \times \frac{u^2\sin^2\theta}{2g}\left[\because H = \frac{u^2\sin^2\theta}{2g}\right]$

$$= \frac{mu^2\sin^2\theta}{2}$$

iv) $(KE)\,\text{top} = \frac{1}{2}mv^2 = \frac{1}{2}m(u\cos\theta)^2\left[\because v = u\cos\theta\right]$

$$= \frac{1}{2}mu^2\cos^2\theta$$

v) $(ME)_{top} = (PE)_{top} + (KE)_{top}$

$$= \cancel{0 +} mgH + \frac{1}{2} m v^2$$

$$= mg \times \frac{u^2 \sin^2 \theta}{2g} + \frac{1}{2} m \times (u \cos \theta)^2$$

$$= \frac{m u^2 \sin^2 \theta}{2} + \frac{m u^2 \cos^2 \theta}{2}$$

$$= \frac{mu^2}{2} \left(\sin^2 \theta + \cos^2 \theta \right)$$

$$= \frac{mu^2}{2} \quad \left[\because \sin^2 \theta + \cos^2 \theta = 1 \right]$$

vi) $\left(\frac{PE}{KE} \right)_{top} = \frac{m u^2 \sin^2 \theta}{2} \div \frac{mu^2 \cos^2 \theta}{2}$

$$= \frac{mu^2 \sin^2 \theta}{2} \times \frac{2}{mu^2 \cos^2 \theta}$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta$$

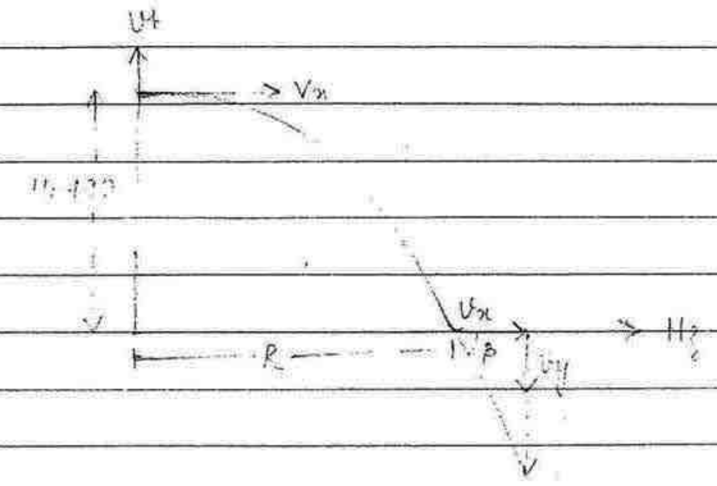
Horizontal Projection:
(PMS) Pg-7

Time of flight, $T = \sqrt{\frac{2H}{g}}$

$R = u \times T = u \sqrt{\frac{2H}{g}}$

equation of Trajectory:
$y = \frac{1}{2} g \left(\frac{x}{u} \right)^2$

Q.15



$H = 490\ m$
$v_x = 98\ m/s$
$g = 9.8\ m/s^2$

i) $T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 490}{9.8}}$
$= \sqrt{\frac{980}{9.8}} = \sqrt{100} = 10\ sec$

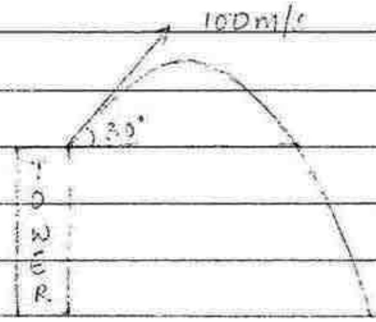
ii) $R = v_x \times T$
$= 98 \times 10 = 980m$

iii) $V = \sqrt{V_x^2 + V_y^2} = \sqrt{u^2 + (gt)^2}$
$= \sqrt{98^2 + (9.8 \times 10)^2}$
$= \sqrt{98^2 + 98^2}$
$= 98\sqrt{2}\ m/s$

for direction,
$\tan \beta = \frac{V_y}{V_x} = \frac{gT}{u}$
$= \frac{9.8 \times 10}{98} = 1 = \tan 45^\circ$

$\therefore \beta = 45^\circ$
$\therefore \vec{V} = 98\sqrt{2}\ m/s$ at an angle of 45° from horizontal.

Q.16 (PMS) Pg-8



$u_x = 100$ m/s

$\theta = 30^\circ$

$H = 120$ m

let downward direction be +ve

$\therefore$ For complete motion,

$S_y = H = 120$ m

$a_y = 10$ m/s^2

$u_y = -u \sin\theta$

$= -100 \sin 30^\circ$

$= -100 \times \frac{1}{2} = -50$ m/s

Using second eq of motion,

$$S_y = u_y t + \frac{1}{2} a_y t^2$$

$$120 = -50t + \frac{1}{2} \times 10 \times t^2$$

$$120 = -50t + 5t^2$$

$$24 = -10t + t^2$$

$$\therefore t^2 - 10t - 24 = 0$$

$$t^2 + 2t - 12t - 24 = 0$$

$$t(t+2) - 12(t+2) = 0.$$

$$t = 12 \quad \text{and} \quad t = -2$$

$\because t = -2$ sec, which is not possible

$\therefore t = 12$ sec