

POLYNOMIALS

Introduction

Algebra is that branch of mathematics which treats the relation of numbers.

Constants And Variables

In algebra, two types of symbols are used : constants and variables (literals).

◆ Constant :

It is a symbol whose value always remains the same, whatever the situation be.

For example : 5, -9, $\frac{3}{8}$, π , $\frac{7}{15}$, etc.

◆ Variable :

It is a symbol whose value changes according to the situation.

For example : x, y, z, ax, a + x, 5y, -7x, etc.

Algebraic expression

- (a) An algebraic expression is a collection of terms separated by plus (+) or minus (-) sign. For example : $3x + 5y$, $7y - 2x$, $2x - ay + az$, etc.
- (b) The various parts of an algebraic expression that are separated by '+' or '-' sign are called terms.

For example :

Algebraic expression	No. of terms	Terms
(i) $-32x$	1	$-32x$
(ii) $2x + 3y$	2	$2x$ and $3y$
(iii) $ax - 5y + cz$	3	ax , $-5y$ and cz
(iv) $\frac{3}{x} + \frac{y}{7} - \frac{xy}{8} + 9$	4	$\frac{3}{x}$, $\frac{y}{7}$, $-\frac{xy}{8}$ and 9 and so on.

Types of Algebraic Expressions :

- (i) **Monomial** : An algebraic expression having only one term is called a monomial. For ex. $8y$, $-7xy$, $4x^2$, abx , etc. 'mono' means 'one'.
- (ii) **Binomial** : An algebraic expression having two terms is called a binomial. For ex. $8x + 3y$, $8x + 3$, $8 + 3y$, $a + bz$, $9 - 4y$, $2x^2 - 4z$, $6y^2 - 5y$, etc. 'bi' means 'two'.
- (iii) **Trinomial** : An algebraic expression having three terms is called a trinomial. For ex. $ax - 5y + 8z$, $3x^2 + 4x + 7$, $9y^2 - 3y + 2x$, etc. 'tri' means 'three'.
- (iv) **Multinomial** : An algebraic expression having two or more terms is called a multinomial.

Factors and Coefficients

◆ Factor :

Each combination of the constants and variables, which form a term, is called a factor.

For examples :

- (i) 7, x and 7x are factors of 7x, in which 7 is constant (numerical) factor and x is variable (literal) factor.
 (ii) In $-5x^2y$, the numerical factor is -5 and literal factors are : x, y, x^2 and x^2y .

◆ **Coefficient :**

Any factor of a term is called the coefficient of the remaining term.

For example :

- (i) In $7x$; 7 is coefficient of x
 (ii) In $-5x^2y$; 5 is coefficient of $-x^2y$; -5 is coefficient of x^2y ,

Ex. 1 Write the coefficient of :

- (i) x^2 in $3x^3 - 5x^2 + 7$
 (ii) xy in $8xyz$
 (iii) $-y$ in $2y^2 - 6y + 2$
 (iv) x^0 in $3x + 7$

Sol. (i) -5 (ii) $8z$ (iii) 6
 (iv) Since $x^0 = 1$, Therefore
 $3x + 7 = 3x + 7x^0$
 \Rightarrow coefficient of x^0 is 7.

A polynomial is an algebraic expression in which each variable involved has power (exponent) a whole number.

For example :

In $3x^2 - y^5 + 8z$, the power of variable x in the term $3x^2$ is 2, the power of variable in the term $-y^5$ is 5 and the power of variable z in the term $8z$ is 1 ($z = z^1$). Therefore, the algebraic expression $3x^2 - y^5 + 8z$ is a polynomial.

◆ **Polynomial in one variable :**

The algebraic expression like $8x$, $7x + 3$, $4y - 3$, $8x^2 + 5$, $6 - z^2$, $x^2 - 5x + 6$, $3y^2 + 8y + 9$, etc. each of which involves only one variable (literal) are called polynomials in one variable.

◆ **Polynomials in two or more variables :**

An algebraic expression, whose term or involves/involve two or more variables (literals) such that the exponent of each variable is a whole number, is called a polynomial in two or more variables.

For examples :

- (a) $3x^2 - 6xy + 8y^2$ is a polynomial in two variables x and y.
 (b) $x + xy^3 - 8x^2yz - 15$ is a polynomial in three variables x, y and z.

Degree of a Polynomial

The greatest power (exponent) of the terms of a polynomial is called degree of the polynomial.

For example :

- (a) In polynomial $5x^2 - 8x^7 + 3x$:
 (i) The power of term $5x^2 = 2$
 (ii) The power of term $-8x^7 = 7$
 (iii) The power of $3x = 1$

Since, the greatest power is 7, therefore degree of the polynomial $5x^2 - 8x^7 + 3x$ is 7

(b) The degree of polynomial :

(i) $4y^3 - 3y + 8$ is 3 (ii) $7p + 2$ is 1 ($p = p^1$)

(iii) $2m - 7m^8 + m^{13}$ is 13 and so on.

◆ EXAMPLES ◆

Ex.2 Find which of the following algebraic expression is a polynomial.

(i) $3x^2 - 5x$ (ii) $x + \frac{1}{x}$

(iii) $\sqrt{y} - 8$ (iv) $z^5 - \sqrt[3]{z} + 8$

Sol. (i) $3x^2 - 5x = 3x^2 - 5x^1$

It is a polynomial.

(ii) $x + \frac{1}{x} = x^1 + x^{-1}$

It is not a polynomial.

(iii) $\sqrt{y} - 8 = y^{1/2} - 8$

Since, the power of the first term (\sqrt{y}) is $\frac{1}{2}$, which is not a whole number.

(iv) $z^5 - \sqrt[3]{z} + 8 = z^5 - z^{1/3} + 8$

Since, the exponent of the second term is $\frac{1}{3}$, which is not a whole number. Therefore, the given expression is not a polynomial.

Ex.3 Find the degree of the polynomial :

(i) $5x - 6x^3 + 8x^7 + 6x^2$

(ii) $2y^{12} + 3y^{10} - y^{15} + y + 3$

(iii) x

(iv) 8

Sol. (i) Since the term with highest exponent (power) is $8x^7$ and its power is 7.

\therefore The degree of given polynomial is 7.

(ii) The highest power of the variable is 15

\Rightarrow degree = 15.

(iii) $x = x^1 \Rightarrow$ degree is 1.

(iv) $8 = 8x^0 \Rightarrow$ degree = 0

Types of Polynomials

(i) Constant Polynomial :

It is a polynomial with degree 0 (zero).

Ex. 20, -8, -1, 1, 5, 7, π , etc.

(ii) Linear Polynomial :

It is a polynomial with degree 1 (one).

Ex. $-8x$, $3x$, x , $x + \sqrt{2}$, $\sqrt{3}x - 2$, $5y - \frac{3}{2}$, $\frac{5}{7}z + 1$ etc.

(iii) Quadratic Polynomial :

It is a polynomial with degree 2 (two)

Ex. $6x^2$, y^2 , $\frac{2}{5}z^2$, $x^2 - 3x$, $x^2 - 3$, $8 - 3y^2$, $5x^2 + 3x + 2$, $7 - 2z + z^2$, etc.

(iv) Cubic Polynomial :

It is a polynomial with degree 3 (three).

Ex. $15y^3$, x^3 , $8z^3$, $x^3 - 5x^2$, $3y^2 + y^3$, $\sqrt{2} + 3z - 2z^2 + 6z^3$, $7y - 2 - 12y^3$, etc.

Identity

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a - b)(a + b) = a^2 - b^2$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$$

$$(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$$

$$(-a + b + c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$$

$$(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$$

$$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$$

$$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$$

$$a^3 + b^3 = (a + b)^3 - 3ab(a + b)$$

$$a^3 - b^3 = (a - b)^3 + 3ab(a - b)$$

$$a^3 + b^3 + c^3 - 3abc$$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

◆ EXAMPLES ◆

Ex.4 Expand each of the following :

(i) $(3x - 4y)^2$ (ii) $\left(\frac{x}{2} + \frac{y}{3}\right)^2$

Sol. (i) We have,

$$\begin{aligned} (3x - 4y)^2 &= (3x)^2 - 2 \times 3x \times 4y + (4y)^2 \\ &= 9x^2 - 24xy + 16y^2 \end{aligned}$$

(ii) We have,

$$\begin{aligned}\left(\frac{x}{2} + \frac{y}{3}\right)^2 &= \left(\frac{x}{2}\right)^2 + 2 \times \frac{x}{2} \times \frac{y}{3} + \left(\frac{y}{3}\right)^2 \\ &= \frac{x^2}{4} + \frac{1}{3}xy + \frac{y^2}{9}\end{aligned}$$

Ex.5 Find the products :

(i) $(2x + 3y)(2x - 3y)$

(ii) $\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

Sol.(i) We have,

$$\begin{aligned}(2x + 3y)(2x - 3y) &= (2x)^2 - (3y)^2 \quad [\text{Using : } (a + b)(a - b) = a^2 - b^2] \\ &= (2x)^2 - (3y)^2 = 4x^2 - 9y^2\end{aligned}$$

(ii) We have,

$$\begin{aligned}\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right) &= \left(x^2 - \frac{1}{x^2}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right) \\ &= \left\{ (x^2)^2 - \left(\frac{1}{x^2}\right)^2 \right\} \left(x^4 + \frac{1}{x^4}\right) \\ &= \left(x^4 - \frac{1}{x^4}\right)\left(x^4 + \frac{1}{x^4}\right) = (x^4)^2 - \left(\frac{1}{x^4}\right)^2 \\ &= x^8 - \frac{1}{x^8}\end{aligned}$$

Ex.6 Evaluate each of the following by using identities :

(i) 103×97 (ii) 103×103 (iii) $(97)^2$

(iv) $185 \times 185 - 115 \times 115$

Sol. (i) We have,

$$\begin{aligned}103 \times 97 &= (100 + 3)(100 - 3) \\ &= (100)^2 - (3)^2 = 10000 - 9 = 9991\end{aligned}$$

(ii) We have,

$$\begin{aligned}103 \times 103 &= (103)^2 \\ &= (100 + 3)^2 = (100)^2 + 2 \times 100 \times 3 + (3)^2 \\ &= 10000 + 600 + 9 = 10609\end{aligned}$$

(iii) We have,

$$\begin{aligned}(97)^2 &= (100 - 3)^2 \\ &= (100)^2 - 2 \times 100 \times 3 + (3)^2 \\ &= 10000 - 600 + 9 = 9409\end{aligned}$$

(iv) We have,

$$\begin{aligned}185 \times 185 - 115 \times 115 &= (185)^2 - (115)^2 = (185 + 115)(185 - 115)\end{aligned}$$

$$= 300 \times 70 = 21000$$

Ex.7 If $x + \frac{1}{x} = 6$, find : $x^4 + \frac{1}{x^4}$

Sol. We have,

$$x^2 + \frac{1}{x^2} = 34 \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = (34)^2$$

$$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right)^2 + 2 \times x^2 \times \frac{1}{x^2} = 1156$$

$$\Rightarrow x^4 + \frac{1}{x^4} + 2 = 1156 \Rightarrow x^4 + \frac{1}{x^4} = 1156 - 2$$

$$\Rightarrow x^4 + \frac{1}{x^4} = 1154$$

Ex.8 If $x^2 + \frac{1}{x^2} = 27$, find the value of the $x - \frac{1}{x}$

Sol. We have,

$$\left(x - \frac{1}{x}\right)^2 = x^2 - 2 \times x \times \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = x^2 - 2 + \frac{1}{x^2}$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 27 - 2 \quad \left[\because x^2 + \frac{1}{x^2} = 27 \text{ (given)}\right]$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 25$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = (\pm 5)^2 \Rightarrow x - \frac{1}{x} = \pm 5$$

Ex.9 If $x + y = 12$ and $xy = 32$, find the value of $x^2 + y^2$.

Sol. We have,

$$(x + y)^2 = x^2 + y^2 + 2xy$$

$$\Rightarrow 144 = x^2 + y^2 + 2 \times 32$$

$$\text{[Putting } x + y = 12 \text{ and } xy = 32]$$

$$\Rightarrow 144 = x^2 + y^2 + 64$$

$$\Rightarrow 144 - 64 = x^2 + y^2$$

$$\Rightarrow x^2 + y^2 = 80$$

Ex. 10 Prove that :

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \\ = [(a - b)^2 + (b - c)^2 + (c - a)^2]$$

Sol. We have,

$$\text{L.H.S.} = 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$$

$$= (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2)$$

$$+ (c^2 - 2ca + a^2) \text{ [Re-arranging the terms]}$$

$$= (a - b)^2 + (b - c)^2 + (c - a)^2 = \text{R.H.S.}$$

$$\text{Hence, } 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$$

$$= [(a - b)^2 + (b - c)^2 + (c - a)^2]$$

Ex.11 If $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Sol. We have,

$$\text{If } a^2 + b^2 + c^2 - ab - bc - ca = 0$$

$$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 2 \times 0$$

[Multiplying both sides by 2]

$$\Rightarrow (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2) = 0$$

$$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$$

$$\Rightarrow a - b = 0, b - c = 0, c - a = 0$$

[\therefore Sum of positive quantities is zero if and only if each quantity is zero]

$$\Rightarrow a = b, b = c \text{ and } c = a$$

$$\Rightarrow a = b = c$$

◆ EXAMPLES ◆

Ex.12 Write the following in expanded form :

(i) $(9x + 2y + z)^2$ (ii) $(3x + 2y - z)^2$

(iii) $(x - 2y - 3z)^2$ (iv) $(-x + 2y + z)^2$

Sol. Using the identity

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

(i) We have,

$$\begin{aligned} (9x + 2y + z)^2 &= (9x)^2 + (2y)^2 + z^2 + 2 \times 9x \times 2y \\ &\quad + 2 \times 2y \times z + 2 \times 9x \times z \\ &= 81x^2 + 4y^2 + z^2 + 36xy + 4yz + 18xz \end{aligned}$$

(ii) We have,

$$\begin{aligned} (3x + 2y - z)^2 &= [3x + 2y + (-z)]^2 \\ &= (3x)^2 + (2y)^2 + (-z)^2 + 2 \times 3x \times 2y \\ &\quad + 2 \times 2y \times (-z) + 2 \times 3x \times (-z) \\ &= 9x^2 + 4y^2 + z^2 + 12xy - 4yz - 6xz \end{aligned}$$

(iii) We have,

$$\begin{aligned} (x - 2y - 3z)^2 &= [x + (-2y) + (-3z)]^2 \\ &= x^2 + (-2y)^2 + (-3z)^2 + 2 \times x \times (-2y) \\ &\quad + 2 \times (-2y) \times (-3z) + 2 \times (-3z) \times x \\ &= x^2 + 4y^2 + 9z^2 - 4xy + 12yz - 6zx \end{aligned}$$

(iv) We have,

$$(-x + 2y + z)^2$$

$$\begin{aligned}
 &= [(-x) + 2y + z]^2 \\
 &= (-x)^2 + (2y)^2 + z^2 + 2 \times (-x) \times (2y) \\
 &\quad + 2 \times 2y \times z + 2 \times (-x) \times z \\
 &= x^2 + 4y^2 + z^2 - 4xy + 4yz - 2zx
 \end{aligned}$$

Ex.13 If $a^2 + b^2 + c^2 = 20$ and $a + b + c = 0$, find $ab + bc + ca$.

Sol. We have,

$$\begin{aligned}
 (a + b + c)^2 &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\
 \Rightarrow (a + b + c)^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\
 \Rightarrow 0^2 &= 20 + 2(ab + bc + ca) \\
 \Rightarrow -20 &= 2(ab + bc + ca) \\
 \Rightarrow -\frac{20}{2} &= \left\{ \frac{2(ab + bc + ca)}{2} \right\} \\
 \Rightarrow ab + bc + ca &= -10
 \end{aligned}$$

Ex.14 If $a + b + c = 9$ and $ab + bc + ca = 40$, find $a^2 + b^2 + c^2$.

Sol. We know that

$$\begin{aligned}
 (a + b + c)^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\
 \Rightarrow 9^2 &= a^2 + b^2 + c^2 + 2 \times 40 \\
 \Rightarrow 81 &= a^2 + b^2 + c^2 + 80 \\
 \Rightarrow a^2 + b^2 + c^2 &= 1
 \end{aligned}$$

Ex.15 If $a^2 + b^2 + c^2 = 250$ and $ab + bc + ca = 3$, find $a + b + c$.

Sol. We know that

$$\begin{aligned}
 (a + b + c)^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\
 \Rightarrow (a + b + c)^2 &= 250 + 2 \times 3 \\
 \Rightarrow (a + b + c)^2 &= 256 \\
 \Rightarrow (a + b + c)^2 &= (\pm 16)^2 \\
 &\quad \text{[Taking square root of both sides]} \\
 \Rightarrow a + b + c &= \pm 16
 \end{aligned}$$

◆ EXAMPLES ◆

Ex.16 Write each of the following in expanded form:

(i) $(2x + 3y)^3$ (ii) $(3x - 2y)^3$

Sol.(i) Replacing a by $2x$ and b by $3y$ in the identity

$$\begin{aligned}
 (a + b)^3 &= a^3 + b^3 + 3ab(a + b), \text{ we have} \\
 (2x + 3y)^3 &= (2x)^3 + (3y)^3 + 3 \times 2x \times 3y \times (2x + 3y) \\
 &= 8x^3 + 27y^3 + 18xy \times 2x + 18xy \times 3y \\
 &= 8x^3 + 27y^3 + 36x^2y + 54xy^2
 \end{aligned}$$

- (ii) Replacing a by $3x$ and b by $2y$ in the identity $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$, we have
- $$(3x - 2y)^3 = (3x)^3 - (2y)^3 - 3 \times 3x \times 2y \times (3x - 2y)$$
- $$= 27x^3 - 8y^3 - 18xy \times (3x - 2y)$$
- $$= 27x^3 - 8y^3 - 54x^2y + 36xy^2$$

Ex.17 If $x + y = 12$ and $xy = 27$, find the value of $x^3 + y^3$.

Sol. We know that

$$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$$

Putting $x + y = 12$ and $xy = 27$ in the above identity, we get

$$12^3 = x^3 + y^3 + 3 \times 27 \times 12$$

$$\Rightarrow 1728 = x^3 + y^3 + 972$$

$$\Rightarrow x^3 + y^3 = 1728 - 972$$

$$\Rightarrow x^3 + y^3 = 756$$

Ex.18 If $x - y = 4$ and $xy = 21$, find the value of $x^3 - y^3$.

Sol. We know that

$$(x - y)^3 = x^3 - y^3 - 3xy(x - y)$$

Putting $x - y = 4$ and $xy = 21$, we get

$$4^3 = x^3 - y^3 - 3 \times 21 \times 4$$

$$\Rightarrow 64 = x^3 - y^3 - 252 \Rightarrow 64 + 252 = x^3 - y^3$$

$$\Rightarrow x^3 - y^3 = 316$$

Ex.19 If $x + \frac{1}{x} = 7$, find the value of $x^3 + \frac{1}{x^3}$.

Sol. We have,

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3 \times x \times \frac{1}{x} \left(x + \frac{1}{x}\right)$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

Putting $x + \frac{1}{x} = 7$, we get

$$7^3 = x^3 + \frac{1}{x^3} + 3 \times 7$$

$$\Rightarrow 343 = x^3 + \frac{1}{x^3} + 21$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 343 - 21 \Rightarrow x^3 + \frac{1}{x^3} = 322$$

Ex.20 If $a + b = 10$ and $a^2 + b^2 = 58$, find the value of $a^3 + b^3$.

Sol. We know that

$$(a + b)^2 = a^2 + b^2 + 2ab$$

Putting $a + b = 10$ and $a^2 + b^2 = 58$, we get

$$10^2 = 58 + 2ab \Rightarrow 100 = 58 + 2ab$$

$$\begin{aligned} \Rightarrow 100 - 58 &= 2ab \Rightarrow 42 = 2ab \\ \Rightarrow ab &= 21 \quad \text{Thus, we have} \\ a + b &= 10 \text{ and } ab = 21 \text{ Now,} \\ (a + b)^3 &= a^3 + b^3 + 3ab(a + b) \\ \Rightarrow 10^3 &= a^3 + b^3 + 3 \times 21 \times 10 \\ &\quad \text{[Putting } a + b = 10 \text{ and } ab = 21\text{]} \\ \Rightarrow 1000 &= a^3 + b^3 + 630 \\ \Rightarrow 1000 - 630 &= a^3 + b^3 \\ \Rightarrow a^3 + b^3 &= 370 \end{aligned}$$

Ex.21 If $x^2 + \frac{1}{x^2} = 7$, find the value of $x^3 + \frac{1}{x^3}$.

Sol. We have,

$$\begin{aligned} \left(x + \frac{1}{x}\right)^2 &= x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x} \\ \Rightarrow \left(x + \frac{1}{x}\right)^2 &= x^2 + \frac{1}{x^2} + 2 \\ \Rightarrow \left(x + \frac{1}{x}\right)^2 &= 7 + 2 \quad \left[\text{Putting } x^2 + \frac{1}{x^2} = 7\right] \\ \Rightarrow \left(x + \frac{1}{x}\right)^2 &= 9 \Rightarrow \left(x + \frac{1}{x}\right) = 3^2 \\ \Rightarrow x + \frac{1}{x} &= 3 \quad \text{[Taking square root of both sides]} \\ \Rightarrow \left(x + \frac{1}{x}\right)^3 &= 3^3 \quad \text{[Cubing both sides]} \\ \Rightarrow x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) &= 27 \\ \Rightarrow \left(x^3 + \frac{1}{x^3}\right) + 3 \times 3 &= 27 \\ \Rightarrow x^3 + \frac{1}{x^3} &= 27 - 9 \Rightarrow x^3 + \frac{1}{x^3} = 18 \end{aligned}$$

Ex.22 If $x^4 + \frac{1}{x^4} = 47$. Find the value of $x^3 + \frac{1}{x^3}$.

Sol. We know that

$$\begin{aligned} \left(x^2 + \frac{1}{x^2}\right)^2 &= x^4 + \frac{1}{x^4} + 2 \times x^2 + \frac{1}{x^2} \\ \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 &= \left(x^4 + \frac{1}{x^4}\right) + 2 \\ \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 &= 47 + 2 \quad \left[\text{Putting } x^4 + \frac{1}{x^4} = 47\right] \\ \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 &= 49 \Rightarrow x^2 + \frac{1}{x^2} = 7 \\ &\quad \text{[Taking square root of both sides]} \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \left(x + \frac{1}{x}\right)^2 &= x^2 + \frac{1}{x^2} + 2 \\
 \Rightarrow \left(x + \frac{1}{x}\right)^2 &= 7 + 2 \quad \left[\text{using : } x^2 + \frac{1}{x^2} = 7\right] \\
 \Rightarrow \left(x + \frac{1}{x}\right)^2 &= 3^2 \Rightarrow x + \frac{1}{x} = 3 \\
 \Rightarrow \left(x + \frac{1}{x}\right)^3 &= 3^3 \quad [\text{Cubing both sides}] \\
 \Rightarrow x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) &= 27 \\
 \Rightarrow x^3 + \frac{1}{x^3} + 3 \times 3 &= 27 \quad \left[\text{Putting } x + \frac{1}{x} = 3\right] \\
 \Rightarrow x^3 + \frac{1}{x^3} &= 27 - 9 \Rightarrow x^3 + \frac{1}{x^3} = 18
 \end{aligned}$$

Ex.23 If $a + b = 10$ and $ab = 21$, find the value of $a^3 + b^3$.

Sol. We know that

$$\begin{aligned}
 a^3 + b^3 &= (a + b)(a^2 - ab + b^2) \\
 &= (a + b)(a^2 + 2ab + b^2 - 2ab - ab) \\
 &[\text{Adding and subtracting } 2ab \text{ in the second bracket}] \\
 &= (a + b)[(a + b)^2 - 3ab] \\
 &= 10 \times (10^2 - 3 \times 21) \\
 &= 10 \times (100 - 63) = 10 \times 37 = 370.
 \end{aligned}$$

Ex.24 If $a - b = 4$ and $ab = 45$, find the value of $a^3 - b^3$.

Sol. We have, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$\begin{aligned}
 &= (a - b)(a^2 - 2ab + b^2 + 2ab + ab) \\
 &= (a - b)\{(a - b)^2 + 3ab\} \\
 &= 4 \times (4^2 + 3 \times 45) = 4 \times (16 + 135) \\
 &= 4 \times 151 = 604.
 \end{aligned}$$

◆ EXAMPLES ◆

Ex.25 If $a + b + c = 0$, then prove that $a^3 + b^3 + c^3 = 3abc$

Sol. We know that

$$\begin{aligned}
 a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\
 \text{putting } a + b + c = 0 \text{ on R.H.S., we get} \\
 a^3 + b^3 + c^3 - 3abc &= 0 \\
 \Rightarrow a^3 + b^3 + c^3 &= 3abc
 \end{aligned}$$

Ex.26 Find the following product :

$$(x + y + 2z)(x^2 + y^2 + 4z^2 - xy - 2yz - 2zx)$$

Sol. We have,

$$(x + y + 2z)(x^2 + y^2 + 4z^2 - xy - 2yz - 2zx)$$

$$= (x + y + 2z)(x^2 + y^2 + (2z)^2 - x \times y - y \times 2z - 2z \times x)$$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca),$$

where $a = x$, $b = y$, $c = 2z$

$$= a^3 + b^3 + c^3 - 3abc$$

$$= x^3 + y^3 + (2z)^3 - 3 \times x \times y \times 2z$$

$$= x^3 + y^3 + 8z^3 - 6xyz$$

Ex.27 If $a + b + c = 6$ and $ab + bc + ca = 11$, find the value of $a^3 + b^3 + c^3 - 3abc$.

Sol. We know that

$$a^3 + b^3 + c^3 - 3abc$$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc =$$

$$(a + b + c) \{(a^2 + b^2 + c^2) - (ab + bc + ca)\} \dots (i)$$

Clearly, we require the values of $a + b + c$, $a^2 + b^2 + c^2$ and $ab + bc + ca$ to obtain the value of $a^3 + b^3 + c^3 - 3abc$. We are given the values of $a + b + c$ and $ab + bc + ca$. So, let us first obtain the value of $a^2 + b^2 + c^2$.

We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\Rightarrow (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$$

$$\Rightarrow 6^2 = a^2 + b^2 + c^2 + 2 \times 11$$

[Putting the values of $a + b + c$ and $ab + bc + ca$]

$$\Rightarrow 36 = a^2 + b^2 + c^2 + 22$$

$$\Rightarrow a^2 + b^2 + c^2 = 36 - 22$$

$$\Rightarrow a^2 + b^2 + c^2 = 14$$

Now, putting $a + b + c = 6$, $ab + bc + ca = 11$ and $a^2 + b^2 + c^2 = 14$ in (i), we get

$$a^3 + b^3 + c^3 - 3abc = 6 \times (14 - 11)$$

$$= 6 \times 3 = 18.$$

Ex.28 If $x + y + z = 1$, $xy + yz + zx = -1$ and $xyz = -1$, find the value of $x^3 + y^3 + z^3$.

Sol. We know that :

$$x^3 + y^3 + z^3 - 3xyz$$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$\Rightarrow x^3 + y^3 + z^3 - 3xyz$$

$$= (x + y + z)(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx - 3xy - 3yz - 3zx)$$

[Adding and subtracting $2xy + 2yz + 2zx$]

$$\Rightarrow x^3 + y^3 + z^3 - 3xyz$$

$$= (x + y + z) \{(x + y + z)^2 - 3(xy + yz + zx)\}$$

$$\Rightarrow x^3 + y^3 + z^3 - 3 \times (-1) = 1 \times \{(1)^2 - 3 \times (-1)\}$$

[Putting the values of $x + y + z$, $xy + yz +$

$$\begin{aligned} & \text{zx and xyz}] \\ \Rightarrow & x^3 + y^3 + z^3 + 3 = 4 \\ \Rightarrow & x^3 + y^3 + z^3 = 4 - 3 \\ \Rightarrow & x^3 + y^3 + z^3 = 1 \end{aligned}$$

Types of Factorization

◆ Type I : Factorization by taking out the common factors.

Ex.29 Factorize the following expression :

$$2x^2y + 6xy^2 + 10x^2y^2$$

Sol. $2x^2y + 6xy^2 + 10x^2y^2 = 2xy(x + 3y + 5xy)$

◆ Type II : Factorization by grouping the terms.

Ex.30 Factorize the following expression :

$$a^2 - b + ab - a$$

Sol. $a^2 - b + ab - a$
 $= a^2 + ab - b - a = (a^2 + ab) - (b + a)$
 $= a(a + b) - (a + b) = (a + b)(a - 1)$

◆ Type III : Factorization by making a perfect square.

Ex.31 Factorize of the following expression :

$$9x^2 + 12xy + 4y^2$$

Sol. $9x^2 + 12xy + 4y^2$
 $= (3x)^2 + 2 \times (3x) \times (2y) + (2y)^2$
 $= (3x + 2y)^2$

Ex.32 Factorize of the following expression :

$$\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2}, x \neq 0, y \neq 0$$

Sol. $\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2}$
 $= \left(\frac{x}{y}\right)^2 + 2 \left(\frac{x}{y}\right)\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = \left(\frac{x}{y} + \frac{y}{x}\right)^2$

Ex.33 Factorize of the following expression

$$\left(5x - \frac{1}{x}\right)^2 + 4 \left(5x - \frac{1}{x}\right) + 4, x \neq 0$$

Sol. $= \left(5x - \frac{1}{x}\right)^2 + 4 \left(5x - \frac{1}{x}\right) + 4$
 $= \left(5x - \frac{1}{x}\right)^2 + 2 \times \left(5x - \frac{1}{x}\right) \times 2 + 2^2$
 $= \left(5x - \frac{1}{x} + 2\right)^2$

◆ Type IV : Factorizing by difference of two squares.

Ex.34 Factorize

(a) $2x^2y + 6xy^2 + 10x^2y^2$

(b) $2x^4 + 2x^3y + 3xy^2 + 3y^3$

Sol. (a) $2x^2y + 6xy^2 + 10x^2y^2$
 $= (2xy)(x + 3y + 5xy)$

(b) $2x^4 + 2x^3y + 3xy^2 + 3y^3$
 $= (2x^4 + 2x^3y) + (3xy^2 + 3y^3)$
 $= (2x^3 + 3y^2)(x + y)$

Ex.35 Factorize $4x^2 + 12xy + 9y^2$

Sol. Note that $4x^2 = (2x)^2 = a^2$ say, and $9y^2 = (3y)^2 = b^2$ say, where $a = 2x$ and $b = 3y$. This suggests the use of identity (i) may be used and the given expression is equal to $(a + b)^2$. Hence

$$\begin{aligned} 4x^2 + 12xy + 9y^2 &= (2x)^2 + 2(x)(3y) + (3y)^2 \\ &= (2x + 3y)^2 \\ &= (2x + 3y)(2x + 3y) \end{aligned}$$

If the expression A can be reduced to an expression, three of whose terms are the squares of some expression, then the identity (vii) may be useful.

Ex.36 Factorize each of the following expressions :

(i) $9x^2 - 4y^2$ (ii) $x^3 - x$

Sol. (i) $9x^2 - 4y^2 = (3x)^2 - (2y)^2$
 $= (3x + 2y)(3x - 2y)$

(ii) $x^3 - x = x(x^2 - 1)$
 $= x(x-1)(x+1)$

Ex.37 Factorize each of the following expressions :

(i) $36x^2 - 12x + 1 - 25y^2$

(ii) $a^2 - \frac{9}{a^2}$, $a \neq 0$

Sol. (i) $36x^2 - 12x + 1 - 25y^2$
 $= (6x)^2 - 2 \times 6x \times 1 + 1^2 - (5y)^2$
 $= (6x - 1)^2 - (5y)^2$
 $= \{(6x - 1) - 5y\} \{(6x - 1) + 5y\}$
 $= (6x - 1 - 5y)(6x - 1 + 5y)$
 $= (6x - 5y - 1)(6x + 5y - 1)$

(ii) $a^2 - \frac{9}{a^2} = (a)^2 - \left(\frac{3}{a}\right)^2$
 $= \left(a - \frac{3}{a}\right) \left(a + \frac{3}{a}\right)$

Ex.38 Factorize the following algebraic expression :

$x^4 - 81y^4$

Sol. $x^4 - 81y^4 = [(x)^2]^2 - (9y^2)^2$
 $= (x^2 - 9y^2)(x^2 + 9y^2)$

$$= \{x^2 - (3y)^2\} (x^2 + 9y^2)$$

$$= (x - 3y) (x + 3y) (x^2 + 9y^2)$$

Ex.39 Factorize the following expression:

$$x(x+z) - y(y+z)$$

Sol.

$$x(x+z) - y(y+z) = (x^2 - y^2) + (xz - yz)$$

$$= (x-y)(x+y) + z(x-y)$$

$$= (x-y)\{(x+y) + z\}$$

$$= (x-y)(x+y+z)$$

Ex.40 Factorize the following expression :

$$x^4 + x^2 + 1$$

Sol.

$$x^4 + x^2 + 1 = (x^4 + 2x^2 + 1) - x^2$$

$$= (x^2 + 1)^2 - x^2$$

$$= (x^2 + 1 - x)(x^2 + 1 + x)$$

$$= (x^2 - x + 1)(x^2 + x + 1)$$

◆ **Type V : Factorizing the sum and difference of cubes of two quantities.**

(i) $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$

(ii) $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$

Ex.41 Factorize the following expression :

$$a^3 + 27$$

Sol.

$$a^3 + 27 = a^3 + 3^3 = (a + 3)(a^2 - 3a + 9)$$

Ex.42 Simplify : $(x+y)^3 - (x-y)^3 - 6y(x^2 - y^2)$

Sol. Let $x + y = a$ and $x - y = b$.

Then, $ab = (x+y)(x-y) = x^2 - y^2$ and

$$a - b = (x+y) - (x-y) = 2y$$

$$\therefore (x+y)^3 - (x-y)^3 - 6y(x^2 - y^2)$$

$$= a^3 - b^3 - 3ab(a - b) = (a - b)^3$$

$$= \{(x+y) - (x-y)\}^3 = (2y)^3 = 8y^3$$

Factorization of the Quadratic polynomial by splitting the middle term

◆ **Type I : Factorization of Quadratic polynomials of the form $x^2 + bx + c$.**

(i) In order to factorize $x^2 + bx + c$ we have to find numbers p and q such that $p + q = b$ and $pq = c$.

(ii) After finding p and q , we split the middle term in the quadratic as $px + qx$ and get desired factors by grouping the terms.

◆ **EXAMPLES** ◆

Ex.43 Factorize each of the following expressions :

(i) $x^2 + 6x + 8$ (ii) $x^2 + 4x - 21$

Sol. (i) In order to factorize $x^2 + 6x + 8$, we find two numbers p and q such that $p + q = 6$ and $pq = 8$.

Clearly, $2 + 4 = 6$ and $2 \times 4 = 8$.

We know split the middle term $6x$ in the given quadratic as $2x + 4x$, so that

$$\begin{aligned}x^2 + 6x + 8 &= x^2 + 2x + 4x + 8 \\&= (x^2 + 2x) + (4x + 8) \\&= x(x + 2) + 4(x + 2) \\&= (x + 2)(x + 4)\end{aligned}$$

(ii) In order to factorize $x^2 + 4x - 21$, we have to find two numbers p and q such that

$$p + q = 4 \text{ and } pq = -21$$

Clearly, $7 + (-3) = 4$ and $7 \times -3 = -21$

We now split the middle term $4x$ of $x^2 + 4x - 21$ as $7x - 3x$, so that

$$\begin{aligned}x^2 + 4x - 21 &= x^2 + 7x - 3x - 21 \\&= (x^2 + 7x) - (3x + 21) \\&= x(x + 7) - 3(x + 7) = (x + 7)(x - 3)\end{aligned}$$

Ex.44 Factorize each of the following quadratic polynomials:

$$x^2 - 21x + 108$$

Sol. In order to factorize $x^2 - 21x + 108$, we have to find two numbers such that their sum is -21 and the product 108 .

Clearly, $-21 = -12 - 9$ and $-12 \times -9 = 108$

$$\begin{aligned}\therefore x^2 - 21x + 108 &= x^2 - 12x - 9x + 108 \\&= (x^2 - 12x) - (9x - 108) \\&= x(x - 12) - 9(x - 12) \\&= (x - 12)(x - 9)\end{aligned}$$

Ex.45 Factorize the following by splitting the middle term : $x^2 + 3\sqrt{3}x + 6$

Sol. In order to factorize $x^2 + 3\sqrt{3}x + 6$, we have to find two numbers p and q such that $p + q = 3\sqrt{3}$ and $pq = 6$

Clearly, $2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$ and $2\sqrt{3} \times \sqrt{3} = 6$

So, we write the middle term $3\sqrt{3}x$ as $2\sqrt{3}x + \sqrt{3}x$, so that

$$\begin{aligned}x^2 + 3\sqrt{3}x + 6 &= x^2 + 2\sqrt{3}x + \sqrt{3}x + 6 \\&= (x^2 + 2\sqrt{3}x) + (\sqrt{3}x + 6) \\&= (x^2 + 2\sqrt{3}x) + (\sqrt{3}x + 2\sqrt{3} \times \sqrt{3}) \\&= x(x + 2\sqrt{3}) + \sqrt{3}(x + 2\sqrt{3}) \\&= (x + 2\sqrt{3})(x + \sqrt{3})\end{aligned}$$

◆ **Type II : Factorization of polynomials reducible to the form $x^2 + bx + c$.**

Ex.46 Factorize : $(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$.

Sol. Let $a^2 - 2a = x$. Then,

$$\begin{aligned}(a^2 - 2a)^2 - 23(a^2 - 2a) + 120 \\&= x^2 - 23x + 120\end{aligned}$$

$$\begin{aligned}\text{Now, } x^2 - 23x + 120 &= x^2 - 15x - 8x + 120 \\&= (x^2 - 15x) - (8x - 120)\end{aligned}$$

$$= x(x - 15) - 8(x - 15)$$

$$= (x - 15)(x - 8)$$

Replacing x by $a^2 - 2a$ on both sides, we get

$$(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$$

$$= (a^2 - 2a - 15)(a^2 - 2a - 8)$$

$$= (a^2 - 5a + 3a - 15)(a^2 - 4a + 2a - 8)$$

$$= \{(a(a - 5) + 3(a - 5))\} \{a(a - 4) + 2(a - 4)\}$$

$$= \{(a - 5)(a + 3)\} \{(a - 4)(a + 2)\}$$

$$= (a - 5)(a + 3)(a - 4)(a + 2)$$

Ex.47 Factorize the following by splitting the middle term :

$$x^4 - 5x^2 + 4$$

Let $x^2 = y$. Then, $x^4 - 5x^2 + 4$

$$= y^2 - 5y + 4$$

Now, $y^2 - 5y + 4$

$$= y^2 - 4y - y + 4$$

$$= (y^2 - 4y) - (y - 4)$$

$$= y(y - 4) - (y - 4)$$

$$= (y - 4)(y - 1)$$

Replacing y by x^2 on both sides, we get

$$x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1)$$

$$= (x^2 - 2^2)(x^2 - 1^2)$$

$$= (x - 2)(x + 2)(x - 1)(x + 1)$$

Ex.48 Factorize : $(x^2 - 4x)(x^2 - 4x - 1) - 20$

Sol. The given expression is

$$(x^2 - 4x)(x^2 - 4x - 1) - 20$$

$$= (x^2 - 4x)^2 - (x^2 - 4x) - 20$$

Let $x^2 - 4x = y$. Then,

$$(x^2 - 4x)^2 - (x^2 - 4x) - 20 = y^2 - y - 20$$

Now, $y^2 - y - 20$

$$= y^2 - 5y + 4y - 20$$

$$= (y^2 - 5y) + (4y - 20)$$

$$= y(y - 5) + 4(y - 5)$$

$$= (y - 5)(y + 4)$$

$$\text{Thus, } y^2 - y - 20 = (y - 5)(y + 4)$$

Replacing y by $x^2 - 4x$ on both sides, we get

$$(x^2 - 4x)^2 - (x^2 - 4x) - 20$$

$$= (x^2 - 4x - 5)(x^2 - 4x + 4)$$

$$= (x^2 - 5x + x - 5)(x^2 - 2 \times x \times 2 + 2^2)$$

$$= \{x(x - 5) + (x - 5)\} (x - 2)^2$$

$$= (x - 5)(x + 1)(x - 2)^2$$

◆ **Type III : Factorization of Expressions which are not quadratic but can be factorized by splitting the middle term.**

Ex.49 If $x^2 + px + q = (x + a)(x + b)$, then factorize $x^2 + pxy + qy^2$.

Sol. We have,

$$x^2 + px + q = (x + a)(x + b)$$

$$\Rightarrow x^2 + px + q = x^2 + x(a + b) + ab$$

On equating the coefficients of like powers of x , we get

$$p = a + b \text{ and } q = ab$$

$$\therefore x^2 + pxy + qy^2 = x^2 + (a + b)xy + aby^2$$

$$= (x^2 + axy) + (bxy + aby^2)$$

$$= x(x + ay) + by(x + ay)$$

$$= (x + ay)(x + by)$$

Ex.50 Factorize the following expression

$$x^2y^2 - xy - 72$$

Sol. In order to factorize $x^2y^2 - xy - 72$, we have to find two numbers p and q such that $p + q = -1$ and $pq = -72$

clearly, $-9 + 8 = -1$ and $-9 \times 8 = -72$. So, we write the middle term $-xy$ of $x^2y^2 - xy - 72$ as $-9xy + 8xy$, so that

$$x^2y^2 - xy - 72 = x^2y^2 - 9xy + 8xy - 72$$

$$= (x^2y^2 - 9xy) + (8xy - 72)$$

$$= xy(xy - 9) + 8(xy - 9)$$

$$= (xy - 9)(xy + 8)$$

Factorization of Polynomials of the Form

$$ax^2 + bx + c, a \neq 0, 1$$

◆ **Type I : Factorization of quadratic polynomials of the form $ax^2 + bx + c, a \neq 0, 1$**

- (i) In order to factorize $ax^2 + bx + c$. We find numbers ℓ and m such that $\ell + m = b$ and $\ell m = ac$
- (ii) After finding ℓ and m , we split the middle term bx as $\ell x + mx$ and get the desired factors by grouping the terms.

◆ EXAMPLES ◆

Ex.51 Factorize the following expression :

$$6x^2 - 5x - 6$$

Sol. The given expression is of the form $ax^2 + bx + c$, where, $a = 6, b = -5$ and $c = -6$.

In order to factorize the given expression, we have to find two numbers ℓ and m such that

$$\ell + m = b = \text{i.e., } \ell + m = -5$$

$$\text{and } \ell m = ac \text{ i.e. } \ell m = 6 \times -6 = -36$$

i.e., we have to find two factors of -36 such that their sum is -5 . Clearly,

$$-9 + 4 = -5 \text{ and } -9 \times 4 = -36$$

$$\therefore \ell = -9 \text{ and } m = 4$$

Now, we split the middle term $-5x$ of

$$\begin{aligned}
 & x^2 - 5x - 6 \text{ as } -9x + 4x, \text{ so that} \\
 & 6x^2 - 5x - 6 = 6x^2 - 9x + 4x - 6 \\
 & = (6x^2 - 9x) + (4x - 6) \\
 & = 3x(2x - 3) + 2(2x - 3) = (2x - 3)(3x + 2)
 \end{aligned}$$

Ex.52 Factorize each of the following expressions :

(i) $\sqrt{3}x^2 + 11x + 6\sqrt{3}$ (ii) $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$
 (iv) $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$

Sol. (i) The given quadratic expression is of the form $ax^2 + bx + c$, where $a = \sqrt{3}$, $b = 11$ and $c = 6\sqrt{3}$.

In order to factorize it, we have to find two numbers ℓ and m such that

$$\ell + m = b = 11 \text{ and } \ell m = ac = \sqrt{3} \times 6\sqrt{3} = 18$$

Clearly, $9 + 2 = 11$ and $9 \times 2 = 18$

$$\therefore \ell = 9 \text{ and } m = 2$$

$$\begin{aligned}
 \text{Now, } & \sqrt{3}x^2 + 11x + 6\sqrt{3} \\
 & = \sqrt{3}x^2 + 9x + 2x + 6\sqrt{3} \\
 & = (\sqrt{3}x^2 + 9x) + (2x + 6\sqrt{3}) \\
 & = (\sqrt{3}x^2 + 3\sqrt{3} \times \sqrt{3}x) + (2x + 6\sqrt{3}) \\
 & = \sqrt{3}x(x + 3\sqrt{3}) + 2(x + 3\sqrt{3}) \\
 & = (\sqrt{3}x + 2)(x + 3\sqrt{3}).
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence, } & \sqrt{3}x^2 + 11x + 6\sqrt{3} \\
 & = (\sqrt{3}x + 2)(x + 3\sqrt{3})
 \end{aligned}$$

(ii) Here, $a = 4\sqrt{3}$, $b = 5$ and $c = -2\sqrt{3}$

In order to factorize $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$, we have to find two numbers ℓ and m such that

$$\begin{aligned}
 \ell + m & = b = 5 \text{ and } \ell m = ac \\
 & = 4\sqrt{3} \times -2\sqrt{3} = -24
 \end{aligned}$$

Clearly, $8 + (-3) = 5$ and $8 \times -3 = -24$

$$\therefore \ell = 8 \text{ and } m = -3$$

$$\begin{aligned}
 \text{Now, } & 4\sqrt{3}x^2 + 5x - 2\sqrt{3} \\
 & = 4\sqrt{3}x^2 + 8x - 3x - 2\sqrt{3} \\
 & = (4\sqrt{3}x^2 + 8x) - (3x + 2\sqrt{3}) \\
 & = 4x(\sqrt{3}x + 2) - \sqrt{3}(\sqrt{3}x + 2) \\
 & = (\sqrt{3}x + 2)(4x - \sqrt{3})
 \end{aligned}$$

(iii) The given quadratic polynomial is $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$.

Clearly, it is of the form $ax^2 + bx + c$, where $a = 7\sqrt{2}$, $b = -10$ and $c = -4\sqrt{2}$.

In order to factorize $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$, we have to find two numbers ℓ and m such that $\ell + m = b = -10$ and

$$\ell m = ac = 7\sqrt{2} \times -4\sqrt{2} = -56$$

Clearly, $-14 + 4 = -10$ and $-14 \times 4 = -56$

$$\therefore \ell = -14 \text{ and } m = 4$$

Now, we split the middle term $-10x$ of $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$ as $-14x + 4x$ so that

$$\begin{aligned} & 7\sqrt{2}x^2 - 10x - 4\sqrt{2} \\ &= 7\sqrt{2}x^2 - 14x + 4x - 4\sqrt{2} \\ &= (7\sqrt{2}x^2 - 14x) + (4x - 4\sqrt{2}) \\ &= (7\sqrt{2}x^2 - 7\sqrt{2} \times \sqrt{2}x) + (4x - 4\sqrt{2}) \\ &= 7\sqrt{2}x(x - \sqrt{2}) + 4(x - \sqrt{2}) \\ &= (x - \sqrt{2})(7\sqrt{2}x + 4) \end{aligned}$$

Ex.53 Factorize the following by splitting the middle term :

$$\frac{1}{3}x^2 - 2x - 9$$

Sol. In order to factorize $\frac{1}{3}x^2 - 2x - 9$, we have to find to number ℓ and m such that

$$\ell + m = -2 \text{ and } \ell m = \frac{1}{3} \times -9 = -3$$

Clearly, $-3 + 1 = -2$ and $-3 \times 1 = -3$

So, we write the middle term $-2x$ as

$-3x + x$, so that

$$\begin{aligned} \frac{1}{3}x^2 - 2x - 9 &= \frac{1}{3}x^2 - 3x + x - 9 \\ &= \left(\frac{1}{3}x^2 - 3x\right) + (x - 9) = \left(\frac{1}{3}x^2 - \frac{9}{3}x\right) + (x - 9) \\ &= (x-9) \left(\frac{1}{3}x + 1\right) \end{aligned}$$

◆ **Type II : Factorization of trinomial expressions which are not quadratic but can be factorized by splitting the middle term.**

Ex.54 Factorize the following trinomial by splitting the middle term :

$$8a^3 - 2a^2b - 15ab^2$$

Sol. Here $a^3 \times ab^2 = (a^2b)^2$ i.e., the product of the variables in first and last term is same as the square of the variables in the middle term. So, in order to factorize the given trinomial, we split the middle term

$-2a^2b$ as $-12a^2b + 10a^2b$, so that

$$\begin{aligned} & 8a^3 - 2a^2b - 15ab^2 \\ &= 8a^3 - 12a^2b + 10a^2b - 15ab^2 \\ &= 4a^2(2a - 3b) + 5ab(2a - 3b) \\ &= (2a - 3b)(4a^2 + 5ab) \\ &= (2a - 3b)a(4a + 5b) \\ &= a(2a - 3b)(4a + 5b) \end{aligned}$$

◆ **Type III : Factorization of trinomial expressions reducible to quadratic expressions.**

Ex. 55 Factorize each of the following expressions by splitting the middle term :

(i) $9(x - 2y)^2 - 4(x - 2y) - 13$

(ii) $2(x + y)^2 - 9(x + y) - 5$

(iii) $8(a + 1)^2 + 2(a + 1)(b + 2) - 15(b + 2)^2$

Sol. (i) The given expression is

$$9(x - 2y)^2 - 4(x - 2y) - 13.$$

Putting $x - 2y = a$, we get

$$9(x - 2y)^2 - 4(x - 2y) - 13 = 9a^2 - 4a - 13$$

Now, $9a^2 - 4a - 13 = 9a^2 - 13a + 9a - 13$

$$= (9a^2 - 13a) + (9a - 13)$$

$$= a(9a - 13) + (9a - 13)$$

$$= (a + 1)(9a - 13)$$

Replacing a by $x - 2y$ on both sides, we get

$$9(x - 2y)^2 - 4(x - 2y) - 13$$

$$= (x - 2y + 1) \{9(x - 2y) - 13\}$$

$$= (x - 2y + 1)(9x - 18y - 13)$$

(ii) The given expression is $2(x + y)^2 - 9(x + y) - 5$ Replacing $x + y$ by a in the given expression, we have

$$2(x + y)^2 - 9(x + y) - 5 = 2a^2 - 9a - 5$$

Now, $2a^2 - 9a - 5 = 2a^2 - 10a + a - 5$

$$= (2a^2 - 10a) + (a - 5)$$

$$= 2a(a - 5) + (a - 5)$$

$$= (a - 5)(2a + 1)$$

Replacing a by $x + y$ on both sides, we get

$$2(x + y)^2 - 9(x + y) - 5$$

$$= (x + y - 5) \{2(x + y) + 1\}$$

$$= (x + y - 5)(2x + 2y + 1).$$

(iii) The given trinomial is

$$8(a + 1)^2 + 2(a + 1)(b + 2) - 15(b + 2)^2$$

Putting $a + 1 = x$ and $b + 2 = y$, we have

$$8(a + 1)^2 + 2(a + 1)(b + 2) - 15(b + 2)^2$$

$$= 8x^2 + 2xy - 15y^2$$

$$= 8x^2 + 12xy - 10xy - 15y^2$$

$$= 4x(2x + 3y) - 5y(2x + 3y)$$

$$= (2x + 3y)(4x - 5y)$$

Replacing x by $a + 1$ and y by $b + 2$, we get

$$8(a + 1)^2 + 2(a + 1)(b + 2) - 15(b + 2)^2$$

$$= \{2(a + 1) + 3(b + 2)\} \{4(a + 1) - 5(b + 2)\}$$

$$= (2a + 3b + 8)(4a - 5b - 6)$$

Factorization of Algebraic expressions of the form $a^3 + b^3 + c^3$, when $a + b + c = 0$ **◆ EXAMPLES ◆****Ex.56** Factorize :

$$(x - y)^3 + (y - z)^3 + (z - x)^3$$

Sol. Let $x - y = a$, $y - z = b$ and $z - x = c$, then,

$$a + b + c = x - y + y - z + z - x = 0$$

$$\therefore a^3 + b^3 + c^3 = 3abc$$

$$\Rightarrow (x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x)$$

Ex.57 Factorize :

$$(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3$$

Sol. We have,

let $x = a^2-b^2$, $y = b^2-c^2$ and $z = c^2-a^2$. Then,

$$x + y + z = a^2-b^2 + b^2-c^2 + c^2-a^2 = 0$$

$$\therefore x^3 + y^3 + z^3 = 3xyz$$

$$\Rightarrow (a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3$$

$$= 3(a^2-b^2)(b^2-c^2)(c^2-a^2)$$

$$= 3(a+b)(a-b)(b+c)(b-c)(c+a)(c-a)$$

$$= 3(a+b)(b+c)(c+a)(a-b)(b-c)(c-a)$$

Ex.58 Simplify : $\frac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3}$

Sol. We have,

$$(a^2-b^2) + (b^2-c^2) + (c^2-a^2) = 0$$

$$\therefore (a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3$$

$$= 3(a^2-b^2)(b^2-c^2)(c^2-a^2)$$

$$= 3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)$$

Similarly,

$$(a-b) + (b-c) + (c-a) = 0$$

$$\Rightarrow (a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a)$$

$$\therefore \frac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3}$$

$$= \frac{3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)}{3(a-b)(b-c)(c-a)}$$

$$= (a+b)(b+c)(c+a)$$

Ex.59 Find the value of $x^3 - 8y^3 - 36xy - 216$, when $x = 2y + 6$.

Sol. We have, $x^3 - 8y^3 - 36xy - 216$

$$= x^3 + (-2y)^3 + (-6)^3 - 3(x)(-2y)(-6)$$

$$= (x - 2y - 6)(x^2 + 4y^2 + 36 + 2xy - 12y + 6x)$$

$$= 0 \times (x^2 + 4y^2 + 36 + 2xy - 12y + 6x)$$

$$[\because x = 2y + 6 \Rightarrow x - 2y - 6 = 0]$$

$$= 0$$

Factorization of $x^3 \pm y^3$

In order to factorize the algebraic expression expressible as the sum or difference of two cubes, we use the following identities.

$$(i) x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$(ii) x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

◆ EXAMPLES ◆

Ex.60 Factorize $27x^3 + 64y^3$

$$\begin{aligned} \text{Sol. } 27x^3 + 64y^3 &= (3x)^3 + (4y)^3 \\ &= (3x + 4y) \{(3x)^2 - (3x)(4y) + (4y)^2\}, \\ &= (3x + 4y) (9x^2 - 12xy + 16y^2) \end{aligned}$$

Ex.61 Factorize $a^3 + 3a^2b + 3ab^2 + b^3 - 8$

$$\begin{aligned} \text{Sol. } a^3 + 3a^2b + 3ab^2 + b^3 - 8 &= (a + b)^3 - 2^3 \\ &= \{(a+b) - 2\} \{(a+b)^2 + (a+b) \cdot 2 + 2^2\} \\ &= (a + b - 2) (a^2 + 2ab + b^2 + 2a + 2b + 4) \end{aligned}$$

Ex.62 Factorize : $a^3 - 0.216$

$$\begin{aligned} \text{Sol. } \text{We have, } a^3 - 0.216 &= a^3 - (0.6)^3 \\ &= (a - 0.6) [a^2 + 0.6a + (0.6)^2] \\ &= (a - 0.6) (a^2 + 0.6a + 0.36) \end{aligned}$$

Ex.63 Factorize :

$$(i) (x+1)^3 - (x-1)^3 \quad (ii) 8(x+y)^3 - 27(x-y)^3$$

$$\begin{aligned} \text{Sol. } (i) (x+1)^3 - (x-1)^3 &= \{(x+1) - (x-1)\} \{(x+1)^2 + (x+1)(x-1) + (x-1)^2\} \\ &= \{(x+1-x+1)\} \{(x^2+2x+1)+(x^2-1) + (x^2 - 2x+1)\} \\ &= 2 (x^2 + 2x + 1 + x^2 - 1 + x^2 - 2x + 1) \\ &= 2 (3x^2+1) \end{aligned}$$

$$\begin{aligned} (ii) \text{ We have, } 8(x+y)^3 - 27(x-y)^3 &= \{2(x+y)\}^3 - \{3(x-y)\}^3 \\ &= \{2(x+y) - 3(x-y)\} [\{2(x+y)\}^2 - 2(x+y) \times 3(x-y) \\ &\quad - 6(x^2 - y^2) + 3(x^2 - 2xy + y^2)] + \{3(x-y)\}^2] \\ &= (2x + 2y - 3x + 3y) \{2(x^2 + 2xy + y^2) \\ &\quad - 6(x^2 - y^2) + 3(x^2 - 2xy + y^2)\} \\ &= (-x + 5y) (2x^2 + 4xy + 2y^2 - 6x^2 + 6y^2 \\ &\quad + 3x^2 - 6xy + 3y^2) \\ &= (-x + 5y) (-x^2 - 2xy + 11y^2) \end{aligned}$$

Ex.64 Factorize : (i) $x^6 - y^6$ (ii) $x^{12} - y^{12}$

$$\begin{aligned} \text{Sol. } (i) \text{ we have, } x^6 - y^6 &= (x^2)^3 - (y^2)^3 = (x^2 - y^2) \{(x^2)^2 + x^2 \cdot y^2 + (y^2)^2\} \\ &= (x^2 - y^2) (x^4 + x^2y^2 + y^4) \\ &= (x-y) (x+y) \{(x^4 + 2x^2y^2 + y^4) - x^2y^2\} \\ &= (x-y) (x+y) \{(x^2 + y^2)^2 - (xy)^2\} \\ &= (x+y) (x-y) \{(x^2 + y^2 - xy)(x^2 + y^2 + xy)\} \\ &= (x+y) (x-y) (x^2 - xy + y^2) (x^2 + xy + y^2) \end{aligned}$$

$$(ii) x^{12} - y^{12} = (x^4)^3 - (y^4)^3$$

$$\begin{aligned}
&= (x^4 - y^4) \{(x^4)^2 + x^4 \times y^4 + (y^4)^2\} \\
&= (x^2)^2 - (y^2)^2 \{(x^8 + x^4 y^4 + y^8)\} \\
&= (x^2 - y^2) (x^2 + y^2) (x^8 + 2x^4 y^4 + y^8 - x^4 y^4) \\
&= (x - y) (x + y) (x^2 + y^2) \{(x^4 + y^4)^2 - (x^2 y^2)^2\} \\
&= (x - y) (x + y) (x^2 + y^2) \{(x^4 + y^4 - x^2 y^2) \\
&\quad (x^4 + y^4 + x^2 y^2)\} \\
&= (x - y) (x + y) (x^2 + y^2) (x^4 + y^4 - x^2 y^2) \{(x^4 + y^4 + 2x^2 y^2) - x^2 y^2\} \\
&= (x - y) (x + y) (x^2 + y^2) (x^4 + y^4 - x^2 y^2) \{(x^2 + y^2)^2 - (xy)^2\} \\
&= (x - y) (x + y) (x^2 + y^2) (x^4 + y^4 - x^2 y^2) (x^2 + y^2 - xy) (x^2 + y^2 + xy) \\
&= (x - y) (x + y) (x^2 + y^2) (x^4 + y^4 - x^2 y^2) (x^2 - xy + y^2) (x^2 + xy + y^2)
\end{aligned}$$

Ex.65 Prove that :

$$\frac{0.87 \times 0.87 \times 0.87 + 0.13 \times 0.13 \times 0.13}{0.87 \times 0.87 - 0.87 \times 0.13 + 0.13 \times 0.13} = 1$$

Sol. We have

$$\begin{aligned}
&\frac{0.87 \times 0.87 \times 0.87 + 0.13 \times 0.13 \times 0.13}{0.87 \times 0.87 - 0.87 \times 0.13 + 0.13 \times 0.13} \\
&= \frac{(0.87)^3 + (0.13)^3}{(0.87)^2 - 0.87 \times 0.13 + (0.13)^2} \\
&= \frac{a^3 + b^3}{a^2 - ab + b^2} \quad \text{where } a = 0.87 \text{ and } b = 0.13 \\
&= \frac{(a + b)(a^2 - ab + b^2)}{(a^2 - ab + b^2)} \\
&= a + b = (0.87 + 0.13) = 1
\end{aligned}$$

Factorization of $x^3 + y^3 + z^3 - 3xyz$

(i) In order to factorize the algebraic expressions of the form $x^3 + y^3 + z^3 - 3xyz$

We use the following identity :

$$\begin{aligned}
&x^3 + y^3 + z^3 - 3xyz \\
&= (x + y + z) (x^2 + y^2 + z^2 - xy - yz - zx)
\end{aligned}$$

(ii) If $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$

◆ EXAMPLES ◆

Ex.66 Factorize : $8x^3 + 27y^3 + z^3 - 18xyz$

Sol. We have,

$$\begin{aligned}
&8x^3 + 27y^3 + z^3 - 18xyz \\
&= (2x)^3 + (3y)^3 + z^3 - 3 \times 2x \times 3y \times z \\
&= (2x + 3y + z) \{(2x)^2 + (3y)^2 + z^2 - 2x \times 3y \\
&\quad - 3y \times z - z \times 2x\} \\
&= (2x + 3y + z) \{(4x^2 + 9y^2 + z^2 - 6xy - 3yz - 2zx)\}
\end{aligned}$$

Ex.67 Factorize :

$$(a+b)^3 + (b+c)^3 + (c+a)^3 - 3(a+b)(b+c)(c+a)$$

Sol. We have,

$$\begin{aligned} & (a+b)^3 + (b+c)^3 + (c+a)^3 - 3(a+b)(b+c)(c+a) \\ &= \{(a+b) + (b+c) + (c+a)\} \{(a+b)^2 + (b+c)^2 + (c+a)^2 - (a+b)(b+c) - (b+c)(c+a) \\ &\quad - (c+a)(a+b)\} \\ &= (2a+2b+2c) \{(a^2 + 2ab + b^2) + (b^2 + 2bc+c^2) \\ &\quad + (c^2 + 2ca + a^2) - (ab + ac+b^2 + bc) \\ &\quad - (bc + ba + c^2 + ca) - (ca + cb + a^2 + ab)\} \\ &= 2(a+b+c) (2a^2 + 2b^2 + 2c^2 + 2ab + 2bc \\ &\quad + 2ca - ab - ac - b^2 - bc - bc - ba - c^2 - ca \\ &\quad - ca - cb - a^2 - ab) \\ &= 2(a+b+c) (a^2 + b^2 + c^2 - ab - bc - ca) \end{aligned}$$

Ex.68 Resolve $a^3 - b^3 + 1 + 3ab$ into factors

Sol.

$$\begin{aligned} & a^3 - b^3 + 1 + 3ab \\ &= a^3 + (-b)^3 + 1^3 - 3(a)(-b)(1) \\ &= (a-b+1)(a^2 + b^2 + 1 + ab - a+b) \\ &= (a-b+1)(a^2 + b^2 + ab - a+b+1) \end{aligned}$$

Ex.69 Factorize : $2\sqrt{2}a^3 + 8b^3 - 27c^3 + 18\sqrt{2}abc$

Sol.

$$\begin{aligned} & 2\sqrt{2}a^3 + 8b^3 - 27c^3 + 18\sqrt{2}abc \\ &= (\sqrt{2}a)^3 + (2b)^3 - (3c)^3 - 3(\sqrt{2}a)(2b)(-3c) \\ &= (\sqrt{2}a + 2b - 3c)(2a^2 + 4b^2 + 9c^2 - 2\sqrt{2}ab \\ &\quad + 6bc + 3\sqrt{2}ac) \end{aligned}$$

Ex.70 Prove that :

$$\begin{aligned} & a^3 + b^3 + c^3 - 3abc \\ &= \frac{1}{2}(a+b+c) \{(a-b)^2 + (b-c)^2 + (c-a)^2\} \end{aligned}$$

Sol. We have,

$$\begin{aligned} & a^3 + b^3 + c^3 - 3abc \\ &= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) \\ &= \frac{1}{2}(a+b+c) \{(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)\} \\ &= \frac{1}{2}(a+b+c) \{(a-b)^2 + (b-c)^2 + (c-a)^2\} \end{aligned}$$

Remainder Theorem and Factor Theorem

- (i) Remainder obtained on dividing polynomial $p(x)$ by $x - a$ is equal to $p(a)$.
- (ii) If a polynomial $p(x)$ is divided by $(x + a)$ the remainder is the value of $p(x)$ at $x = -a$.

- (iii) If a polynomial $p(x)$ is divided by $(ax - b)$, the remainder is the value of $p(x)$ at $x = \frac{b}{a}$
- (iv) If a polynomial $p(x)$ is divided by $(b - ax)$, the remainder is equal to the value of $p(x)$ at $x = \frac{b}{a}$.
- (v) $(x - a)$ is a factor of polynomial $p(x)$ if $p(a) = 0$
- (vi) $(x + a)$ is a factor of polynomial $p(x)$ if $p(-a) = 0$
- (vii) $(ax - b)$ is a factor of polynomial $p(x)$ if $p\left(\frac{b}{a}\right) = 0$.
- (viii) $(x - a)(x - b)$ is a factor of polynomial $p(x)$, if $p(a) = 0$ and $p(b) = 0$.

◆ EXAMPLES ◆

Ex.71 Find the remainder when $4x^3 - 3x^2 + 2x - 4$ is divided by

(a) $x - 1$ (b) $x + 2$ (c) $x + \frac{1}{2}$

Sol. Let $p(x) = 4x^3 - 3x^2 + 2x - 4$

(a) When $p(x)$ is divided by $(x - 1)$, then by remainder theorem, the required remainder will be $p(1)$

$$\begin{aligned} p(1) &= 4(1)^3 - 3(1)^2 + 2(1) - 4 \\ &= 4 \times 1 - 3 \times 1 + 2 \times 1 - 4 \\ &= 4 - 3 + 2 - 4 = -1 \end{aligned}$$

(b) When $p(x)$ is divided by $(x + 2)$, then by remainder theorem, the required remainder will be $p(-2)$.

$$\begin{aligned} p(-2) &= 4(-2)^3 - 3(-2)^2 + 2(-2) - 4 \\ &= 4 \times (-8) - 3 \times 4 - 4 - 4 \\ &= -32 - 12 - 8 = -52 \end{aligned}$$

(c) When $p(x)$ is divided by, $\left(x + \frac{1}{2}\right)$ then by remainder theorem, the required remainder will be

$$\begin{aligned} p\left(-\frac{1}{2}\right) &= 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) - 4 \\ &= 4 \times \left(-\frac{1}{8}\right) - 3 \times \frac{1}{4} - 2 \times \frac{1}{2} - 4 \\ &= -\frac{1}{2} - \frac{3}{4} - 1 - 4 = -\frac{1}{2} - \frac{3}{4} - 5 \\ &= \frac{-2 - 3 - 20}{4} = \frac{-25}{4} \end{aligned}$$

Ex.72 Determine the remainder when the polynomial $p(x) = x^4 - 3x^2 + 2x + 1$ is divided by $x - 1$.

Sol. By remainder theorem, the required remainder is equal to $p(1)$.

$$\begin{aligned} \text{Now, } p(x) &= x^4 - 3x^2 + 2x + 1 \\ \Rightarrow p(1) &= (1)^4 - 3 \times 1^2 + 2 \times 1 + 1 \\ &= 1 - 3 + 2 + 1 = 1 \end{aligned}$$

Hence required remainder = $p(1) = 1$

Ex.73 Find the remainder when the polynomial $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$ is divided by $x + 2$

Sol. We have, $x + 2 = x - (-2)$. So, by remainder theorem, when $f(x)$ is divided by $(x - (-2))$ the remainder is equal to $f(-2)$.

$$\text{Now, } f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$$

$$\Rightarrow f(-2) = 2(-2)^4 - 6(-2)^3 + 2(-2)^2 - (-2) + 2$$

$$\Rightarrow f(-2) = 2 \times 16 - 6 \times -8 + 2 \times 4 + 2 + 2$$

$$\Rightarrow f(-2) = 32 + 48 + 8 + 2 + 2 = 92$$

Hence, required remainder = 92

Ex.74 Find the remainder when

$$p(x) = 4x^3 - 12x^2 + 14x - 3 \text{ is divided by } g(x) = x - \frac{1}{2}$$

Sol. By remainder theorem, we know that $p(x)$ when divided by $g(x) = \left(x - \frac{1}{2}\right)$ gives a remainder equal to $p\left(\frac{1}{2}\right)$.

$$\text{Now, } p(x) = 4x^3 - 12x^2 + 14x - 3$$

$$\Rightarrow p\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 14\left(\frac{1}{2}\right) - 3$$

$$\Rightarrow p\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{12}{4} + \frac{14}{2} - 3$$

$$\Rightarrow p\left(\frac{1}{2}\right) = \frac{1}{2} - 3 + 7 - 3$$

$$\Rightarrow p\left(\frac{1}{2}\right) = \frac{3}{2}$$

Hence, required remainder = $p\left(\frac{1}{2}\right) = \frac{3}{2}$

Ex.75 If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x-3)$, find the value of a .

Sol. Let $p(x) = ax^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + a$ be the given polynomials. The remainders when $p(x)$ and $q(x)$ are divided by $(x-3)$ are $p(3)$ and $q(3)$ respectively.

By the given condition, we have

$$p(3) = q(3)$$

$$\Rightarrow a \times 3^3 + 4 \times 3^2 + 3 \times 3 - 4 = 3^3 - 4 \times 3 + a$$

$$\Rightarrow 27a + 36 + 9 - 4 = 27 - 12 + a$$

$$\Rightarrow 26a + 26 = 0$$

$$\Rightarrow 26a = -26$$

$$\Rightarrow a = -1$$

Ex.76 Let R_1 and R_2 are the remainders when the polynomials $x^3 + 2x^2 - 5ax - 7$ and

$x^3 + ax^2 - 12x + 6$ are divided by $x + 1$ and $x - 2$ respectively. If $2R_1 + R_2 = 6$, find the value of a .

Sol. Let $p(x) = x^3 + 2x^2 - 5ax - 7$ and

$q(x) = x^3 + ax^2 - 12x + 6$ be the given polynomials.

Now, $R_1 =$ Remainder when $p(x)$ is divided by $x + 1$

$$\Rightarrow R_1 = p(-1)$$

$$\Rightarrow R_1 = (-1)^3 + 2(-1)^2 - 5a \times -1 - 7$$

$$[\because p(x) = x^3 + 2x^2 - 5ax - 7]$$

$$\Rightarrow R_1 = -1 + 2 + 5a - 7$$

$$\Rightarrow R_1 = 5a - 6$$

And, $R_2 =$ Remainder when $q(x)$ is divided by $x - 2$

$$\Rightarrow R_2 = q(2)$$

$$\Rightarrow R_2 = (2)^3 + a \times 2^2 - 12 \times 2 + 6$$

$$[\because q(x) = x^3 + ax^2 - 12x + 6]$$

$$\Rightarrow R_2 = 8 + 4a - 24 + 6$$

$$\Rightarrow R_2 = 4a - 10$$

Substituting the values of R_1 and R_2 in $2R_1 + R_2 = 6$, we get

$$2(5a - 6) + (4a - 10) = 6$$

$$\Rightarrow 10a - 12 + 4a - 10 = 6$$

$$\Rightarrow 14a - 22 = 6$$

$$\Rightarrow 14a = 28$$

$$\Rightarrow a = 2$$

Factor Theorem

Step 1 : $(x + a)$ is factor of a polynomial $p(x)$ if $p(-a) = 0$.

Step 2 : $(ax - b)$ is a factor of a polynomial $p(x)$ if $p(b/a) = 0$

Step 3 : $ax + b$ is a factor of a polynomial $p(x)$ if $p(-b/a) = 0$.

Step 4 : $(x - a)(x - b)$ is a factor of a polynomial $p(x)$ if $p(a) = 0$ and $p(b) = 0$

◆ EXAMPLES ◆

Ex.77 Use the factor theorem to determine whether $x - 1$ is a factor of

(a) $x^3 + 8x^2 - 7x - 2$

(b) $2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$

(c) $8x^4 + 12x^3 - 18x + 14$

Sol.(a) Let $p(x) = x^3 + 8x^2 - 7x - 2$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$

$$p(1) = (1)^3 + 8(1)^2 - 7(1) - 2$$

$$= 1 + 8 - 7 - 2$$

$$= 9 - 9 = 0$$

Hence $(x - 1)$ is a factor of $p(x)$.

(b) Let $p(x) = 2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$.

$$p(1) = 2\sqrt{2}(1)^3 + 5\sqrt{2}(1)^2 - 7\sqrt{2}$$

$$= 2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2}$$

$$= 7\sqrt{2} - 7\sqrt{2} = 0$$

Hence $(x-1)$ is a factor of $p(x)$

(c) Let $p(x) = 8x^4 + 12x^3 - 18x + 14$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$

$$p(1) = 8(1)^4 + 12(1)^3 - 18(1) + 14$$

$$= 8 + 12 - 18 + 14$$

$$= 34 - 18 = 16 \neq 0.$$

Hence $(x-1)$ is not a factor of $p(x)$.

Ex.78 Factorize each of the following expression, given that ;

$x^3 + 13x^2 + 32x + 20$. $(x+2)$ is a factor.

Sol. First method

Let $p(x) = x^3 + 13x^2 + 32x + 20$

$$= (x+2) \text{ is a factor of } p(x)$$

$$p(x) = (x+2)(x^2 + 11x + 10)$$

$$= (x+2)(x^2 + 10x + x + 10)$$

$$= (x+2)(x+10)(x+1)$$

Ex.79 Factorize ;

$$x^3 - 23x^2 + 142x - 120$$

Sol. Let $p(x) = x^3 - 23x^2 + 142x - 120$

Constant term, $p(x)$ is -120

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm 12, \dots, \pm 120$

$$P(1) = 1 - 23 + 142 - 120 = 0$$

$\Rightarrow x - 1$ is a factor of $p(x)$. We find the other factor by dividing $p(x)$ by $(x-1)$

$$p(x) = (x-1)(x^2 - 22x + 120)$$

$$= (x-1)(x^2 - 10x - 12x + 120)$$

$$= (x-1)[x(x-10) - 12(x-10)]$$

$$= (x-1)(x-10)(x-12)$$

Ex.80 Show that $(x - 3)$ is a factor of the polynomial $x^3 - 3x^2 + 4x - 12$

Sol. Let $p(x) = x^3 - 3x^2 + 4x - 12$ be the given polynomial. By factor theorem, $(x - a)$ is a factor of a polynomial $p(x)$ iff $p(a) = 0$. Therefore, in order to prove that $x - 3$ is a factor of $p(x)$, it is sufficient to show that $p(3) = 0$. Now,

$$p(x) = x^3 - 3x^2 + 4x - 12$$

$$\Rightarrow p(3) = 3^3 - 3 \times 3^2 + 4 \times 3 - 12$$

$$= 27 - 27 + 12 - 12 = 0$$

Hence, $(x - 3)$ is a factor of

$$p(x) = x^3 - 3x^2 + 4x - 12.$$

Ex.81 Show that $(x - 1)$ is a factor of $x^{10} - 1$ and also of $x^{11} - 1$.

Sol. Let $f(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$.

In order to prove that $(x - 1)$ is a factor of

both $f(x)$ and $g(x)$, it is sufficient to show that $f(1) = 0$ and $g(1) = 0$.

$$\text{Now, } f(x) = x^{10} - 1 \quad \text{and } g(x) = x^{11} - 1$$

$$\Rightarrow f(1) = 1^{10} - 1 = 0 \quad \text{and } g(1) = 1^{11} - 1 = 0$$

$$\Rightarrow (x - 1) \text{ is a factor of both } f(x) \text{ and } g(x)$$

Ex.82 Show that $x + 1$ and $2x - 3$ are factors of $2x^3 - 9x^2 + x + 12$.

Sol. Let $p(x) = 2x^3 - 9x^2 + x + 12$ be the given polynomial. In order to prove that $x + 1$ and $2x - 3$ are factors of $p(x)$, it is sufficient to show that $p(-1)$ and $p(3/2)$ both are equal to zero.

$$\text{Now, } p(x) = 2x^3 - 9x^2 + x + 12$$

$$\Rightarrow p(-1) = 2 \times (-1)^3 - 9 \times (-1)^2 + (-1) + 12$$

$$\text{and, } p\left(\frac{3}{2}\right) = 2 \times \left(\frac{3}{2}\right)^3 - 9 \times \left(\frac{3}{2}\right)^2 + \frac{3}{2} + 12$$

$$\Rightarrow p(-1) = -2 - 9 - 1 + 12 \text{ and}$$

$$p\left(\frac{3}{2}\right) = \frac{54}{8} - \frac{81}{4} + \frac{3}{2} + 12$$

$$\Rightarrow p(-1) = -12 + 12 \text{ and}$$

$$p\left(\frac{3}{2}\right) = \frac{54 - 162 + 12 + 96}{8}$$

$$\Rightarrow p(-1) = 0 \quad \text{and } p\left(\frac{3}{2}\right) = 0$$

Hence, $(x + 1)$ and $(3x - 2)$ are factors of the given polynomial.

Ex.83 Find the value of k , if $x + 3$ is a factor of $3x^2 + kx + 6$.

Sol. Let $p(x) = 3x^2 + kx + 6$ be the given polynomial. Then, $(x + 3)$ is a factor of $p(x)$

$$\Rightarrow p(-3) = 0$$

$$\Rightarrow 3(-3)^2 + k \times (-3) + 6 = 0$$

$$\Rightarrow 27 - 3k + 6 = 0$$

$$\Rightarrow 33 - 3k = 0 \Rightarrow k = 11$$

Hence, $x + 3$ is a factor of $3x^2 + kx + 6$ if $k = 11$.

Ex.84 If $ax^3 + bx^2 + x - 6$ has $x + 2$ as a factor and leaves a remainder 4 when divided by $(x - 2)$, find the values of a and b .

Sol. Let $p(x) = ax^3 + bx^2 + x - 6$ be the given polynomial. Then, $(x + 2)$ is a factor of $p(x)$

$$\Rightarrow p(-2) = 0 \quad [\because x + 2 = 0 \Rightarrow x = -2]$$

$$\Rightarrow a(-2)^3 + b(-2)^2 + (-2) - 6 = 0$$

$$\Rightarrow -8a + 4b - 2 - 6 = 0 \Rightarrow -8a + 4b = 8$$

$$\Rightarrow -2a + b = 2 \quad \dots(i)$$

It is given that $p(x)$ leaves the remainder 4 when it is divided by $(x - 2)$. Therefore,

$$p(2) = 4 \quad [\because x - 2 = 0 \Rightarrow x = 2]$$

$$\Rightarrow a(2)^3 + b(2)^2 + 2 - 6 = 4$$

$$\Rightarrow 8a + 4b - 4 = 4 \Rightarrow 8a + 4b = 8$$

$$\Rightarrow 2a + b = 2 \quad \dots(ii)$$

Adding (i) and (ii), we get

$$2b = 4 \Rightarrow b = 2$$

Putting $b = 2$ in (i), we get

$$-2a + 2 = 2 \Rightarrow -2a = 0 \Rightarrow a = 0.$$

Hence, $a = 0$ and $b = 2$.

Ex.85 If both $x - 2$ and $x - \frac{1}{2}$ are factors of $px^2 + 5x + r$, show that $p = r$.

Sol. Let $f(x) = px^2 + 5x + r$ be the given polynomial. Since $x - 2$ and $x - \frac{1}{2}$ are factors of $f(x)$.

Therefore,

$$f(2) = 0 \text{ and } f\left(\frac{1}{2}\right) = 0$$

$$\left[\because x - 2 = 0 \Rightarrow x = 2 \text{ and } x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2} \right]$$

$$\Rightarrow p \times 2^2 + 5 \times 2 + r = 0 \text{ and}$$

$$p\left(\frac{1}{2}\right)^2 + 5 \times \frac{1}{2} + r = 0$$

$$\Rightarrow 4p + 10 + r = 0 \text{ and } \frac{p}{4} + \frac{5}{2} + r = 0$$

$$\Rightarrow 4p + r = -10 \text{ and } \frac{p + 4r + 10}{4} = 0$$

$$\Rightarrow 4p + r = -10 \text{ and } p + 4r + 10 = 0$$

$$\Rightarrow 4p + r = -10 \text{ and } p + 4r = -10$$

$$\Rightarrow 4p + r = p + 4r$$

[\because RHS of the two equations are equal]

$$\Rightarrow 3p = 3r \Rightarrow p = r$$

Ex.86 If $x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, show that $a + c + e = b + d = 0$.

Sol. Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ be the given polynomial. Then, $(x^2 - 1)$ is a factor of $p(x)$

$$\Rightarrow (x - 1)(x + 1) \text{ is a factor of } p(x)$$

$$\Rightarrow (x - 1) \text{ and } (x + 1) \text{ are factors of } p(x)$$

$$\Rightarrow p(1) = 0 \text{ and } p(-1) = 0$$

$$[\because x - 1 = 0 \Rightarrow x = 1 \text{ and } x + 1 = 0 \Rightarrow x = -1]$$

$$\Rightarrow a + b + c + d + e = 0 \text{ and } a - b + c - d + e = 0$$

Adding and subtracting these two equations, we get $2(a + c + e) = 0$ and $2(b + d) = 0$

$$\Rightarrow a + c + e = 0 \text{ and } b + d = 0$$

$$\Rightarrow a + c + e = b + d = 0$$

Ex.87 Using factor theorem, show that $a - b$, $b - c$ and $c - a$ are the factors of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

Sol. By factor theorem, $a - b$ will be a factor of the given expression if it vanishes by substituting $a = b$ in it.

Substituting $a = b$ in the given expression, we have

$$\begin{aligned} & a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2) \\ &= b(b^2 - c^2) + b(c^2 - b^2) + c(b^2 - b^2) \\ &= b^3 - bc^2 + bc^2 - b^3 + c(b^2 - b^2) = 0 \end{aligned}$$

$\therefore (a - b)$ is a factor of

$$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2).$$

Similarly, we can show that $(b - c)$ and $(c - a)$ are also factors of the given expression.

Hence, $(a - b)$, $(b - c)$ and $(c - a)$ are also factors of the given expression.

Application of remainder theorem in the factorization of polynomials

- (i) Obtain the polynomial $p(x)$
- (ii) Obtain the constant term in $p(x)$ and find its all possible factors. For example, in the polynomial $x^4 + x^3 - 7x^2 - x + 6$ the constant term is 6 and its factors are $\pm 1, \pm 2, \pm 3, \pm 6$.
- (iii) Take one of the factors, say a and replace x by it in the given polynomial. If the polynomial reduces to zero, then $(x - a)$ is a factor of polynomial.
- (iv) Obtain the factors equal in no. to the degree of polynomial. Let these are $(x - a), (x - b), (x - c), \dots$
- (v) Write $p(x) = k(x - a)(x - b)(x - c) \dots$ where k is constant.
- (vi) Substitute any value of x other than a, b, c, \dots and find the value of k .

◆ EXAMPLES ◆

Ex.88 Factorize $x^2 + 4 + 9z^2 + 4x - 6xz - 12z$

Sol. The presence of the three squares viz. x^2 , $(2)^2$, and $(3z)^2$ gives a clue that identity (vii) could be used. So we write.

$$A = x^2 + (2)^2 + (3z)^2 + 4x - 6xz - 12z$$

We note that the last two of the product terms are negative and that both of these contain z . Hence we write A as

$$\begin{aligned} A &= x^2 + (2)^2 + (-3z)^2 + 2.2x - 2.x.(-3z) + 2.2(-3z) &&= (x+2-3z)^2 \\ &= (x+2-3z)(x+2-3z) \end{aligned}$$

Ex.89 Using factor theorem, factorize the polynomial $x^3 - 6x^2 + 11x - 6$.

Sol. Let $f(x) = x^3 - 6x^2 + 11x - 6$

The constant term in $f(x)$ is equal to -6 and factors of -6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

Putting $x = 1$ in $f(x)$, we have

$$\begin{aligned} f(1) &= 1^3 - 6 \times 1^2 + 11 \times 1 - 6 \\ &= 1 - 6 + 11 - 6 = 0 \end{aligned}$$

$\therefore (x - 1)$ is a factor of $f(x)$

Similarly, $x - 2$ and $x - 3$ are factors of $f(x)$.

Since $f(x)$ is a polynomial of degree 3. So, it can not have more than three linear factors.

Let $f(x) = k(x - 1)(x - 2)(x - 3)$. Then,

$$x^3 - 6x^2 + 11x - 6 = k(x - 1)(x - 2)(x - 3)$$

Putting $x = 0$ on both sides, we get

$$-6 = k(0-1)(0-2)(0-3)$$

$$\Rightarrow -6 = -6k \Rightarrow k = 1$$

Putting $k = 1$ in $f(x) = k(x-1)(x-2)(x-3)$, we get

$$f(x) = (x-1)(x-2)(x-3)$$

$$\text{Hence, } x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$$

Ex.90 Using factor theorem, factorize the polynomial $x^4 + x^3 - 7x^2 - x + 6$

Sol. Let $f(x) = x^4 + x^3 - 7x^2 - x + 6$

the factors of constant term in $f(x)$ are $\pm 1, \pm 2, \pm 3$ and ± 6

Now,

$$f(1) = 1 + 1 - 7 - 1 + 6 = 8 - 8 = 0$$

$\Rightarrow (x-1)$ is a factor of $f(x)$

$$f(-1) = 1 - 1 - 7 + 1 + 6 = 8 - 8 = 0$$

$\Rightarrow x+1$ is a factor of $f(x)$

$$f(2) = 2^4 + 2^3 - 7 \times 2^2 - 2 + 6$$

$$= 16 + 8 - 28 - 2 + 6 = 0$$

$\Rightarrow x-2$ is a factor of $f(x)$

$$f(-2) = (-2)^4 + (-2)^3 - 7(-2)^2 - (-2) + 6$$

$$= 16 - 8 - 28 + 2 + 6 = -12 \neq 0$$

$\Rightarrow x+2$ is not a factor of $f(x)$

$$f(-3) = (-3)^4 + (-3)^3 - 7(-3)^2 - (-3) + 6$$

$$= 81 - 27 - 63 + 3 + 6 = 90 - 90 = 0$$

$\Rightarrow x+3$ is a factor of $f(x)$

Since $f(x)$ is a polynomial of degree 4. So, it cannot have more than 4 linear factors

Thus, the factors of $f(x)$ are $(x-1), (x+1), (x-2)$ and $(x+3)$.

$$\text{Let } f(x) = k(x-1)(x+1)(x-2)(x+3)$$

$$\Rightarrow x^4 + x^3 - 7x^2 - x + 6 = k(x-1)(x+1)(x-2)(x+3)$$

Putting $k = 0$ on both sides, we get

$$6 = k(-1)(1)(-2)(3) \Rightarrow 6 = 6k \Rightarrow k = 1$$

Substituting $k = 1$ in (i), we get

$$x^4 + x^3 - 7x^2 - x + 6 = (x-1)(x+1)(x-2)(x+3)$$

Ex.91 Factorize, $2x^4 + x^3 - 14x^2 - 19x - 6$

Sol. Let $f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ be the given polynomial. The factors of the constant term -6 are $\pm 1, \pm 2, \pm 3$ and ± 6 , we have,

$$f(-1) = 2(-1)^4 + (-1)^3 - 14(-1)^2 - 19(-1) - 6$$

$$= 2 - 1 - 14 + 19 - 6 = 21 - 21 = 0$$

and,

$$f(-2) = 2(-2)^4 + (-2)^3 - 14(-2)^2 - 19(-2) - 6$$

$$= 32 - 8 - 56 + 38 - 6 = 0$$

So, $x+1$ and $x+2$ are factors of $f(x)$.

$\Rightarrow (x + 1)(x + 2)$ is also a factor of $f(x)$

$\Rightarrow x^2 + 3x + 2$ is a factor of $f(x)$

Now, we divide

$f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ by

$x^2 + 3x + 2$ to get the other factors.

$$\begin{array}{r}
 2x^2 - 5x - 3 \\
 x^2 + 3x + 2 \quad 2x^4 + x^3 - 14x^2 - 19x - 6 \\
 \underline{2x^4 + 6x^3 + 4x^2} \\
 -5x^3 - 18x^2 - 19x - 6 \\
 \underline{-5x^3 - 15x^2 - 10x} \\
 -3x^2 - 9x - 6 \\
 \underline{-3x^2 - 9x - 6} \\
 0
 \end{array}$$

$$\begin{aligned}
 \therefore 2x^4 + x^3 - 14x^2 - 19x - 6 \\
 &= (x^2 + 3x + 2)(2x^2 - 5x - 3) \\
 &= (x + 1)(x + 2)(2x^2 - 5x - 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } 2x^2 - 5x - 3 &= 2x^2 - 6x + x - 3 \\
 &= 2x(x - 3) + 1(x - 3) \\
 &= (x - 3)(2x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence, } 2x^4 + x^3 - 14x^2 - 19x - 6 \\
 &= (x + 1)(x + 2)(x - 3)(2x + 1)
 \end{aligned}$$

Ex.92 Factorize, $9z^3 - 27z^2 - 100z + 300$, if it is given that $(3z+10)$ is a factor of it.

Sol. Let us divide $9z^3 - 27z^2 - 100z + 300$ by $3z + 10$ to get the other factors

$$\begin{array}{r}
 3z^2 - 19z + 30 \\
 3z + 10 \quad 9z^3 - 27z^2 - 100z + 300 \\
 \underline{9z^3 + 30z^2} \\
 -57z^2 - 100z + 300 \\
 \underline{-57z^2 - 190z} \\
 90z + 300 \\
 \underline{90z + 300} \\
 0
 \end{array}$$

$$\therefore 9z^3 - 27z^2 - 100z + 300$$

$$\begin{aligned}
 &= (3z + 10) (3z^2 - 19z + 30) \\
 &= (3z + 10) (3z^2 - 10z - 9z + 30) \\
 &= (3z + 10) \{ (3z^2 - 10z) - (9z - 30) \} \\
 &= (3z + 10) \{ z(3z - 10) - 3(3z - 10) \} \\
 &= (3z + 10) (3z - 10) (z - 3)
 \end{aligned}$$

Hence, $9z^3 - 27z^2 - 100z + 300$

$$= (3z + 10) (3z - 10) (z - 3)$$

Ex.93 Simplify: $\frac{4x-2}{x^2-x-2} + \frac{3}{2x^2-7x+6} - \frac{8x+3}{2x^2-x-3}$

Sol. $\frac{2(2x-1)}{(x-2)(x+1)} + \frac{3}{(2x-3)(x-2)} - \frac{8x+3}{(2x-3)(x+1)}$

The L.C.M. of the factors in the denominator is $(x-2)(x+1)(2x-3)$

The given expression can be reduced to

$$\begin{aligned}
 &\frac{2(2x-1)(2x-3) + 3(x+1) - (8x+3)(x-2)}{(x-2)(x+1)(2x-3)} \\
 &= \frac{2(4x^2 - 8x + 3) + 3(x+1) - (8x^2 - 13x - 6)}{(x-2)(x+1)(2x-3)} \\
 &= \frac{15}{(x-2)(x+1)(2x-3)}
 \end{aligned}$$

Ex.94 Establish the identity

$$\frac{6x^2+11x-8}{3x-2} = (2x+5) + \frac{2}{3x-2}$$

Sol. $\frac{3x-2}{3x-2} \cdot \frac{6x^2+11x-8}{3x-2} = \frac{8(2x+5)}{3x-2} + \frac{2}{3x-2}$

$$\frac{6x^2-4x}{3x-2}$$

$$\frac{15x-8}{3x-2}$$

$$\frac{15x-10}{3x-2}$$

$$\frac{2}{3x-2}$$

$$\therefore \frac{6x^2+11x-8}{3x-2} = (2x+5) + \frac{2}{3x-2}$$

Ex.95 If $\left(x + \frac{1}{x}\right) = 5$, find the value of

$$\left(x^3 + \frac{1}{x^3}\right) - 5 \left[x^2 + \frac{1}{x^2}\right] + 5$$

Sol. $x + \frac{1}{x} = 5$

Squaring, $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2x \left[\frac{1}{x}\right] = 25$

$$\therefore x^2 + \frac{1}{x^2} + 2 = 25.$$

$$\text{Hence } x^2 + \frac{1}{x^2} = 25 - 2 = 23 \quad \dots(i)$$

We know that $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

$$\begin{aligned} \therefore \left(x + \frac{1}{x}\right)^3 &= x^3 + \frac{1}{x^3} + 3x \left(\frac{1}{x}\right) \left[x + \frac{1}{x}\right] \\ &= 5^3 = 125 \end{aligned}$$

$$\text{Hence } x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = 125$$

$$\text{i.e. } x^3 + \frac{1}{x^3} + 3(5) = 125$$

$$\text{or } x^3 + \frac{1}{x^3} = 125 - 15 = 110$$

From (i) and (ii)

$$\begin{aligned} \left[x^3 + \frac{1}{x^3}\right] - 5\left[x^2 + \frac{1}{x^2}\right] + 5 \\ = 110 - 5(23) + 5 \\ = 110 - 115 + 5 = 0 \end{aligned}$$

Values of a Polynomial

For a polynomial $f(x) = 3x^2 - 4x + 2$.

To find its value at $x = 3$;
replace x by 3 everywhere.

So, the value of $f(x) = 3x^2 - 4x + 2$ at $x = 3$ is

$$\begin{aligned} f(3) &= 3 \times 3^2 - 4 \times 3 + 2 \\ &= 27 - 12 + 2 = 17. \end{aligned}$$

Similarly, the value of polynomial

$$f(x) = 3x^2 - 4x + 2,$$

$$\begin{aligned} \text{(i) at } x = -2 \text{ is } f(-2) &= 3(-2)^2 - 4(-2) + 2 \\ &= 12 + 8 + 2 = 22 \end{aligned}$$

$$\begin{aligned} \text{(ii) at } x = 0 \text{ is } f(0) &= 3(0)^2 - 4(0) + 2 \\ &= 0 - 0 + 2 = 2 \end{aligned}$$

$$\begin{aligned} \text{(iii) at } x = \frac{1}{2} \text{ is } f\left(\frac{1}{2}\right) &= 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 2 \\ &= \frac{3}{4} - 2 + 2 = \frac{3}{4} \end{aligned}$$

Ex.96 Find the value of the polynomial $5x - 4x^2 + 3$ at:

$$\text{(i) } x = 0 \quad \text{(ii) } x = -1$$

Sol. Let $p(x) = 5x - 4x^2 + 3$.

$$\begin{aligned} \text{(i) At } x = 0, p(0) &= 5 \times 0 - 4 \times (0)^2 + 3 \\ &= 0 - 0 + 3 = 3 \end{aligned}$$

$$\begin{aligned} \text{(ii) At } x = -1, p(-1) &= 5(-1) - 4(-1)^2 + 3 \\ &= -5 - 4 + 3 = -6 \end{aligned}$$

Zeros of a Polynomial

If for $x = a$, the value of the polynomial $p(x)$ is 0 i.e., $p(a) = 0$; then $x = a$ is a zero of the polynomial

$p(x)$.

For example :

(i) For polynomial $p(x) = x - 2$; $p(2) = 2 - 2 = 0$
 $\therefore x = 2$ or simply 2 is a zero of the polynomial
 $p(x) = x - 2$.

(ii) For the polynomial $g(u) = u^2 - 5u + 6$;
 $g(3) = (3)^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$
 $\therefore 3$ is a zero of the polynomial $g(u)$
 $= u^2 - 5u + 6$.

Also, $g(2) = (2)^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

$\therefore 2$ is also a zero of the polynomial
 $g(u) = u^2 - 5u + 6$

- (a) Every linear polynomial has one and only one zero.
 (b) A given polynomial may have more than one zeroes.
 (c) If the degree of a polynomial is n ; the largest number of zeroes it can have is also n .

For example :

If the degree of a polynomial is 5, the polynomial can have at the most 5 zeroes; if the degree of a polynomial is 8; largest number of zeroes it can have is 8.

(d) A zero of a polynomial need not be 0.

For example : If $f(x) = x^2 - 4$,
 then $f(2) = (2)^2 - 4 = 4 - 4 = 0$

Here, zero of the polynomial $f(x) = x^2 - 4$ is 2 which itself is not 0.

(e) 0 may be a zero of a polynomial.

For example : If $f(x) = x^2 - x$,
 then $f(0) = 0^2 - 0 = 0$

Here 0 is the zero of polynomial $f(x) = x^2 - x$.

◆ EXAMPLES ◆

Ex.97 Verify whether the indicated numbers are zeroes of the polynomial corresponding to them in the following cases :

- (i) $p(x) = 3x + 1$, $x = -\frac{1}{3}$
 (ii) $p(x) = (x + 1)(x - 2)$, $x = -1, 2$
 (iii) $p(x) = x^2$, $x = 0$
 (iv) $p(x) = \ell x + m$, $x = -\frac{m}{\ell}$
 (v) $p(x) = 2x + 1$, $x = \frac{1}{2}$

Sol. (i) $p(x) = 3x + 1$

$$\Rightarrow p\left(-\frac{1}{3}\right) = 3 \times -\frac{1}{3} + 1 = -1 + 1 = 0$$

$\therefore x = -\frac{1}{3}$ is a zero of $p(x) = 3x + 1$.

(ii) $p(x) = (x + 1)(x - 2)$

$$\Rightarrow p(-1) = (-1 + 1)(-1 - 2) = 0 \times -3 = 0$$

and, $p(2) = (2 + 1)(2 - 2) = 3 \times 0 = 0$

$\therefore x = -1$ and $x = 2$ are zeroes of the given polynomial.

(iii) $p(x) = x^2 \Rightarrow p(0) = 0^2 = 0$

$\therefore x = 0$ is a zero of the given polynomial

(iv) $p(x) = \ell x + m \Rightarrow p\left(-\frac{m}{\ell}\right) = \ell\left(-\frac{m}{\ell}\right) + m$
 $= -m + m = 0$

$\therefore x = -\frac{m}{\ell}$ is a zero of the given polynomial.

(v) $P(x) = 2x + 1 \Rightarrow p\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} + 1$
 $= 1 + 1 = 2 \neq 0$

$\therefore x = \frac{1}{2}$ is not a zero of the given polynomial.

Ex.98 Find the zero of the polynomial in each of the following cases :

(i) $p(x) = x + 5$ (ii) $p(x) = 2x + 5$

(iii) $p(x) = 3x - 2$

Sol. To find the zero of a polynomial $p(x)$ means to solve the polynomial equation $p(x) = 0$.

(i) For the zero of polynomial $p(x) = x + 5$

$$p(x) = 0 \Rightarrow x + 5 = 0 \Rightarrow x = -5$$

$\therefore x = -5$ is a zero of the polynomial $p(x) = x + 5$.

(ii) $p(x) = 0 \Rightarrow 2x + 5 = 0$

$$\Rightarrow 2x = -5 \text{ and } x = -\frac{5}{2}$$

$\therefore x = -\frac{5}{2}$ is a zero of $p(x) = 2x + 5$.

(iii) $p(x) = 0 \Rightarrow 3x - 2 = 0$

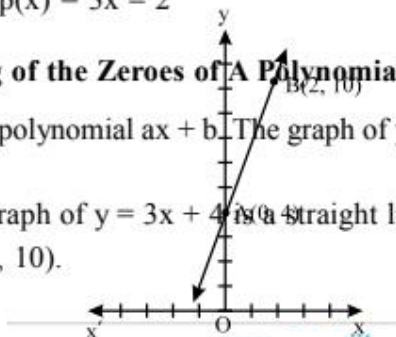
$$\Rightarrow 3x = 2 \text{ and } x = \frac{2}{3}$$

$\therefore x = \frac{2}{3}$ is zero of $p(x) = 3x - 2$

Geometric meaning of the Zeroes of A Polynomial

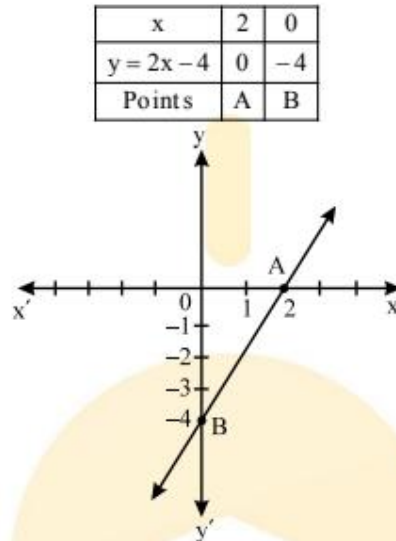
Let us consider linear polynomial $ax + b$. The graph of $y = ax + b$ is a straight line.

For example : The graph of $y = 3x + 4$ is a straight line passing through $(0, 4)$ and $(2, 10)$.



x	0	2
$y = 3x + 4$	4	10
Points	A	B

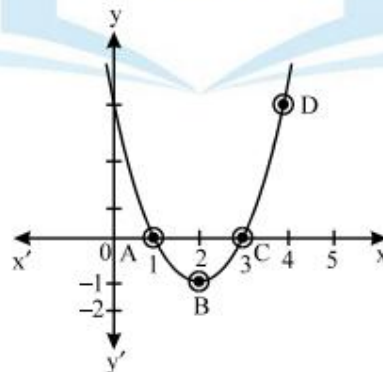
- (i) Let us consider the graph of $y = 2x - 4$ intersects the x-axis at $x = 2$. The zero $2x - 4$ is 2. Thus, the zero of the polynomial $2x - 4$ is the x-coordinate of the point where the graph $y = 2x - 4$ intersects the x-axis.



- (ii) A general equation of a linear polynomial is $ax + b$. The graph of $y = ax + b$ is a straight line which intersects the x-axis at $\left(\frac{-b}{a}, 0\right)$.
Zero of the polynomial $ax + b$ is the x-coordinate of the point of intersection of the graph with x-axis.
- (iii) Let us consider the quadratic polynomial $x^2 - 4x + 3$. The graph of $x^2 - 4x + 3$ intersects the x-axis at the point $(1, 0)$ and $(3, 0)$. Zeroes of the polynomial $x^2 - 4x + 3$ are the x-coordinates of the points of intersection of the graph with x-axis.

x	1	2	3	4	5
$y = x^2 - 4x + 3$	0	-1	0	3	8
Points	A	B	C	D	E

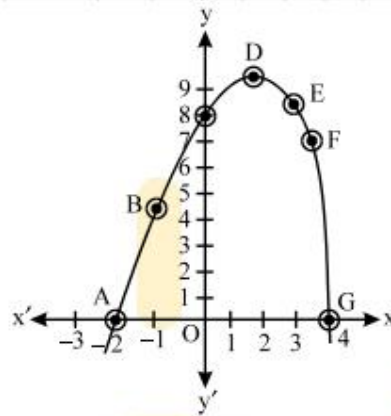
The shape of the graph of the quadratic polynomials is \cup and the curve is known as parabola.



- (iv) Now let us consider one more polynomial $-x^2 + 2x + 8$. Graph of this polynomial intersects the x-axis at the points $(4, 0)$ $(-2, 0)$.
Zeroes of the polynomial $-x^2 + 2x + 8$ are the

x-coordinates of the points at which the graph intersects the x-axis. The shape of the graph of the given quadratic polynomial is \cap and the curve is known as parabola.

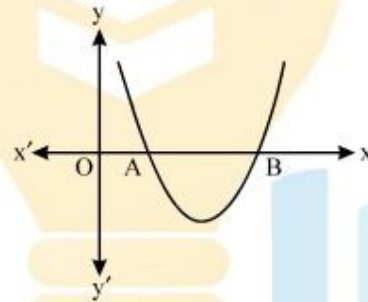
x	-2	-1	0	1	2	3	4
y	0	5	8	9	8	7	0
Points	A	B	C	D	E	F	G



The zeroes of a quadratic polynomial $ax^2 + bx + c$, $a \neq 0$ are the x-coordinates of the points where the graph of $y = ax^2 + bx + c$ intersects the x-axis. There are three types of the graph of $y = ax^2 + bx + c$.

Case I :

Graph of $y = ax^2 + bx + c$ intersects the x-axis at two distinct points A and B. The zeroes of the quadratic polynomial $ax^2 + bx + c$ are the x-coordinates of the points A and B.



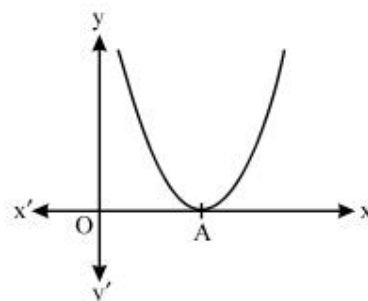
Condition : $b^2 - 4ac > 0$ and $a > 0$

For example : Quadratic polynomial $x^2 - 7x + 12$

Graph of $y = x^2 - 7x + 12$ will cut x-axis at the two distinct points (3, 0) and (4, 0). Zeroes of the polynomial are 3 and 4.

Case II :

Here the graph intersects the x-axis at exactly one point i.e., at two coincident points. These two coincident points A and B coincide and becomes one point A. Zero of the quadratic polynomial is the x-coordinate of point A.



Condition : $b^2 - 4ac = 0$ and $a > 0$

For example : $y = (x - 1)^2$

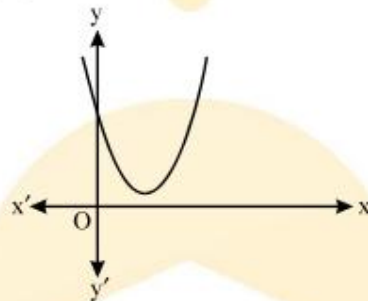
The graph of $y = (x - 1)^2$ will cut x-axis at one point (1, 0). Zero of the polynomial of the point of intersection with x-axis.

Case III :

Here the graph of the quadratic equation will not cut the x-axis. Either the graph will be completely above the x-axis or below the x-axis. So the quadratic polynomial $ax^2 + bx + c$ has no zero in this case.

For example : $y = x^2 - 2x + 4$

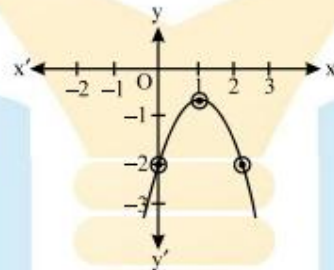
Graph of $y = x^2 - 2x + 4$ will not intersect the x-axis and the graph will be above the x-axis. The polynomial $x^2 - 2x + 4$ has no zero.



Let $y = -x^2 + 2x - 2$

Graph of $y = -x^2 + 2x - 2$ will not intersect the x-axis and the graph will be below the x-axis.

The polynomial $-x^2 + 2x - 2$ has no zero.



In Brief : It means that a polynomial of degree two has at most two zeroes.

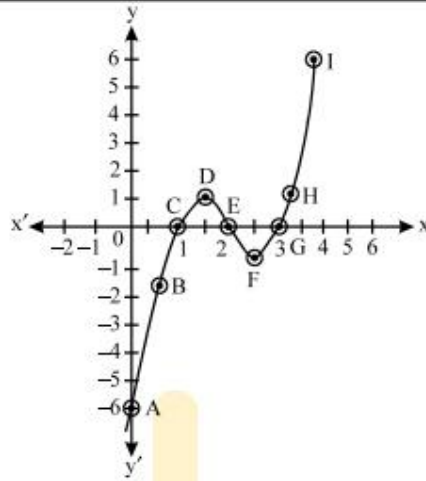
Cubic polynomial : Let us find out geometrically how many zeroes a cubic has.

Let consider cubic polynomial $x^3 - 6x^2 + 11x - 6$.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$y = x^3 - 6x^2 + 11x - 6$	-6	-1.875	0	0.375	0	-0.375	0	1.875	6
Points	A	B	C	D	E	F	G	H	I

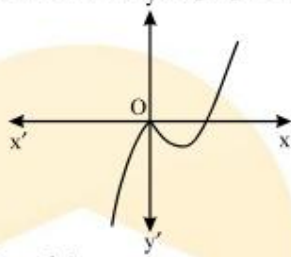
Case 1 :

The graph of the cubic equation intersects the x-axis at three points (1, 0), (2, 0) and (3, 0). Zeroes of the given polynomial are the x-coordinates of the points of intersection with the x-axis.



Case 2 :

The cubic equation $x^3 - x^2$ intersects the x-axis at the point (0, 0) and (1, 0). Zero of a polynomial $x^3 - x^2$ are the x-coordinates of the point where the graph cuts the x-axis.

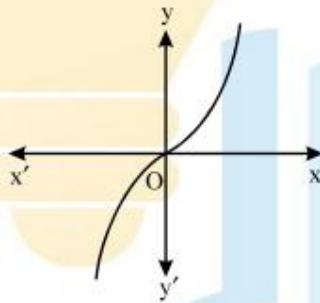


Zeroes of the cubic polynomial are 0 and 1.

Case 3 :

$$y = x^3$$

Cubic polynomial has only one zero.

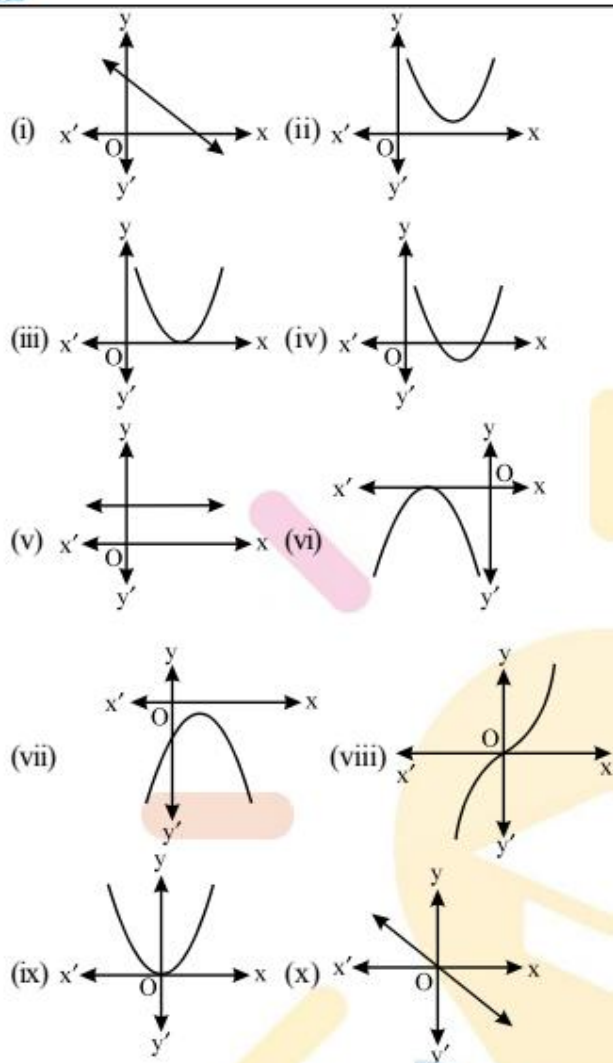


In brief : A cubic equation can have 1 or 2 or 3 zeroes or any polynomial of degree three can have at most three zeroes.

Remarks : In general, polynomial of degree n, the graph of $y = p(x)$ passes x-axis at most at n points. Therefore, a polynomial $p(x)$ of degree n has at most n zeroes.

◆ **EXAMPLES** ◆

Ex.99 Which of the following correspond to the graph to a linear or a quadratic polynomial and find the number of zeroes of polynomial.



- Sol.**
- (i) The graph is a straight line so the graph is of a linear polynomial. The number of zeroes is one as the graph intersects the x-axis at one point only.
 - (ii) The graph is a parabola. So, this is the graph of quadratic polynomial. The number of zeroes is zero as the graph does not intersect the x-axis.
 - (iii) Here the polynomial is quadratic as the graph is a parabola. The number of zeroes is one as the graph intersects the x-axis at one point only (two coincident points).
 - (iv) Here, the polynomial is quadratic as the graph is a parabola. The number of zeroes is two as the graph intersects the x-axis at two points.
 - (v) The polynomial is linear as the graph is straight line. The number of zeroes is zero as the graph does not intersect the x-axis.
 - (vi) The polynomial is quadratic as the graph is a parabola. The number of zeroes is 1 as the graph intersects the x-axis at one point (two coincident points) only.
 - (vii) The polynomial is quadratic as the graph is a parabola. The number of zeroes is zero, as the graph does not intersect the x-axis.
 - (viii) Polynomial is neither linear nor quadratic as the graph is neither a straight line nor a parabola is one as the graph intersects the x-axis at one point only.

- (ix) Here, the polynomial is quadratic as the graph is a parabola. The number of zeroes is one as the graph intersects the x-axis at one point only (two coincident points).
- (x) The polynomial is linear as the graph is a straight line. The number of zeroes is one as the graph intersects the x-axis at only one point.

Relationship Between the Zeroes and the Coefficients of a Polynomial.

Consider quadratic polynomial $P(x) = 2x^2 - 16x + 30$.

$$\begin{aligned} \text{Now, } 2x^2 - 16x + 30 &= (2x - 6)(x - 3) \\ &= 2(x - 3)(x - 5) \end{aligned}$$

The zeroes of $P(x)$ are 3 and 5.

Sum of the zeroes

$$= 3 + 5 = 8 = \frac{-(-16)}{2} = -\left[\frac{\text{coefficient of } (x)}{\text{coefficient of } x^2}\right]$$

Product of the zeroes

$$= 3 \times 5 = 15 = \frac{30}{2} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

◆ EXAMPLES ◆

Ex.100 Find the zeroes of the quadratic polynomial $6x^2 - 13x + 6$ and verify the relation between the zeroes and its coefficients.

Sol. We have, $6x^2 - 13x + 6 = 6x^2 - 4x - 9x + 6$
 $= 2x(3x - 2) - 3(3x - 2)$
 $= (3x - 2)(2x - 3)$

So, the value of $6x^2 - 13x + 6$ is 0, when $(3x - 2) = 0$ or $(2x - 3) = 0$ i.e.,

$$\text{When } x = \frac{2}{3} \text{ or } \frac{3}{2}$$

Therefore, the zeroes of $6x^2 - 13x + 6$ are

$$\frac{2}{3} \text{ and } \frac{3}{2}.$$

Sum of the zeroes

$$= \frac{2}{3} + \frac{3}{2} = \frac{13}{6} = \frac{-(-13)}{6} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.101 Find the zeroes of the quadratic polynomial $4x^2 - 9$ and verify the relation between the zeroes and its coefficients.

Sol. We have,
 $4x^2 - 9 = (2x)^2 - 3^2 = (2x - 3)(2x + 3)$

So, the value of $4x^2 - 9$ is 0, when

$$2x - 3 = 0 \quad \text{or} \quad 2x + 3 = 0$$

$$\text{i.e., when } x = \frac{3}{2} \text{ or } x = -\frac{3}{2}.$$

Therefore, the zeroes of $4x^2 - 9$ are $\frac{3}{2}$ and $-\frac{3}{2}$.

Sum of the zeroes

$$= \frac{3}{2} - \frac{3}{2} = 0 = \frac{-(0)}{4} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \left(\frac{3}{2}\right) \left(-\frac{3}{2}\right) = \frac{-9}{4} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.102 Find the zeroes of the quadratic polynomial $9x^2 - 5$ and verify the relation between the zeroes and its coefficients.

Sol. We have,

$$9x^2 - 5 = (3x)^2 - (\sqrt{5})^2 = (3x - \sqrt{5})(3x + \sqrt{5})$$

So, the value of $9x^2 - 5$ is 0,

$$\text{when } 3x - \sqrt{5} = 0 \text{ or } 3x + \sqrt{5} = 0$$

$$\text{i.e., when } x = \frac{\sqrt{5}}{3} \text{ or } x = \frac{-\sqrt{5}}{3}.$$

Sum of the zeroes

$$= \frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3} = 0 = \frac{-(0)}{9} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \left(\frac{\sqrt{5}}{3}\right) \left(\frac{-\sqrt{5}}{3}\right) = \frac{-5}{9} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.103 If α and β are the zeroes of $ax^2 + bx + c$, $a \neq 0$ then verify the relation between the zeroes and its coefficients.

Sol. Since α and β are the zeroes of polynomial $ax^2 + bx + c$.

Therefore, $(x - \alpha)$, $(x - \beta)$ are the factors of the polynomial $ax^2 + bx + c$.

$$\Rightarrow ax^2 + bx + c = k(x - \alpha)(x - \beta)$$

$$\Rightarrow ax^2 + bx + c = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$$

$$\Rightarrow ax^2 + bx + c = kx^2 - k(\alpha + \beta)x + k\alpha\beta \dots (i)$$

Comparing the coefficients of x^2 , x and constant terms of (1) on both sides, we get

$$a = k, \quad b = -k(\alpha + \beta) \text{ and } c = k\alpha\beta$$

$$\Rightarrow \alpha + \beta = -\frac{b}{k} \text{ and } \alpha\beta = \frac{c}{k}$$

$$\alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a} \quad [\because k = a]$$

$$\text{Sum of the zeroes} = \frac{-b}{a} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

$$\text{Product of the zeroes} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Aliter : Relation between the zeroes and the coefficient of the quadratic polynomial $ax^2 + bx + c$.

Sol. Let α and β be the zeroes of the polynomial $ax^2 + bx + c$

$$\therefore \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \dots(1)$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \dots(2)$$

By adding (1) and (2), we get

$$\begin{aligned} \alpha + \beta &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2b}{2a} = -\frac{b}{a} = \frac{\text{coefficient of } x}{\text{coefficient of } x^2} \end{aligned}$$

Hence, sum of the zeroes of the polynomial

$$ax^2 + bx + c \text{ is } -\frac{b}{a}$$

By multiplying (1) and (2), we get

$$\begin{aligned} \alpha\beta &= \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) \\ &= \frac{(-b)^2 - \sqrt{(b^2 - 4ac)^2}}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2} \\ &= \frac{4ac}{4a^2} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2} \end{aligned}$$

$$\text{Hence, product of zeroes} = \frac{c}{a}$$

In general, it can be proved that if α, β, γ are the zeroes of a cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

Note, $\frac{b}{a}, \frac{c}{a}$ and $\frac{d}{a}$ are meaningful because $a \neq 0$.

Ex.104 Find the zeroes of the quadratic polynomial

$x^2 - 2x - 8$ and verify a relationship between zeroes and its coefficients.

Sol.

$$x^2 - 2x - 8 = x^2 - 4x + 2x - 8$$

$$= x(x - 4) + 2(x - 4) = (x - 4)(x + 2)$$

So, the value of $x^2 - 2x - 8$ is zero when $x - 4 = 0$ or $x + 2 = 0$ i.e., when $x = 4$ or $x = -2$.

So, the zeroes of $x^2 - 2x - 8$ are 4, -2.

Sum of the zeroes

$$= 4 - 2 = 2 = \frac{-(-2)}{1} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= 4(-2) = -8 = \frac{-8}{1} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.105 Find the zeroes of the quadratic polynomial

$4s^2 - 4s + 1$ and verify the relationship between zeroes and its coefficients.

Sol. We have,

$$\begin{aligned} 4s^2 - 4s + 1 &= 4s^2 - 2s - 2s + 1 \\ &= 2s(2s - 1) - 1(2s - 1) \\ &= (2s - 1)(2s - 1) = (2s - 1)^2 \end{aligned}$$

So, the value of $4s^2 - 4s + 1$ is zero when $2s - 1 = 0$, or $s = \frac{1}{2}$.

Zeros of the polynomial are $\frac{1}{2}, \frac{1}{2}$

Sum of the zeroes

$$= \frac{1}{2} + \frac{1}{2} = 1 = -\left(\frac{-4}{4}\right) = \frac{-\text{coefficient of } s}{\text{coefficient of } s^2}$$

Product of the zeroes

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} = \frac{\text{constant term}}{\text{coefficient of } s^2}$$

Ex.106 Verify that the number given along sides of the cubic polynomials are their zeroes. Also verify the relationship between the zeroes and the coefficients, $x^3 + 2x^2 - x - 2$; 1, -1, -2.

Sol. Here the polynomial $p(x)$ is

$$x^3 + 2x^2 - x - 2$$

Value of the polynomial $x^3 + 2x^2 - x - 2$ when $x = 1$

$$= (1)^3 + 2(1)^2 - 1 - 2 = 1 + 2 - 1 - 2 = 0$$

So, 1 is a zero of $p(x)$.

On putting $x = -1$ in the cubic polynomial

$$x^3 + 2x^2 - x - 2.$$

$$= (-1)^3 + 2(-1)^2 - (-1) - 2 = -1 + 2 + 1 - 2 = 0$$

So, -1 is a zero of $p(x)$.

On putting $x = -2$ in the cubic polynomial

$$\begin{aligned} x^3 + 2x^2 - x + 2 &= (-2)^3 + 2(-2)^2 - 2 + 2 \\ &= -8 + 8 + 2 - 2 = 0 \end{aligned}$$

So, -2 is a zero of $p(x)$.

Hence 1, -1 and -2 are the zeroes of the given polynomial.

Sum of the zeroes of $p(x)$

$$= 1 - 1 - 2 = \frac{-2}{1} = \frac{-\text{coefficient of } x^2}{\text{coefficient of } x^3}$$

Sum of the products of two zeroes taken at a time

$$= (1)(-1) + (-1)(-2) + (1)(-2) = -1 + 2 - 2 = -1$$

$$= \frac{-1}{1} = \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$$

Product of all three zeroes

$$= (1)(-1)(-2) = 2 = \frac{-(-2)}{1} = -\frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.107 Verify that the number given along the sides of the cubic polynomials are their zeroes. Also verify the relationship between the zeroes and the coefficients. $x^3 - 4x^2 + 5x - 2$; 2, 1, 1.

Sol. Here the polynomial $p(x)$ is

$$x^3 - 4x^2 + 5x - 2$$

Value of the polynomial $x^3 - 4x^2 + 5x - 2$,

when $x = 2$

$$= (2)^3 - 4(2)^2 + 5(2) - 2 = 8 - 16 + 10 - 2 = 0$$

So, 2 is a zero of $p(x)$.

On putting $x = 1$ in the cubic polynomial

$$x^3 - 4x^2 + 5x - 2$$

$$= (1)^3 - 4(1)^2 + 5(1) - 2$$

$$= 1 - 4 + 5 - 2 = 0$$

So, 1 is a zero of $p(x)$.

Hence, 2, 1, 1 are the zeroes of the given polynomial.

Sum of the zeroes of $p(x) = 2 + 1 + 1 = 4$

$$= \frac{-(-4)}{1} = \frac{-\text{coefficient of } x^2}{\text{coefficient of } x^3}$$

Sum of the products of two zeroes taken at a time

$$= 2 \times 1 + 1 \times 1 + 2 \times 1 = 2 + 1 + 2$$

$$= \frac{5}{1} = \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$$

Product of all the three zeroes

$$= 2 \times 1 \times 1 = 2 = \frac{-(-2)}{1} = \frac{-(\text{constant term})}{\text{coefficient of } x^2}$$

Ex.108 Verify that the numbers given along side of the cubic polynomials are their zeroes. Also verify the relationship between the zeroes and the coefficients. $2x^3 + x^2 - 5x + 2$; $\frac{1}{2}, 1, -2$

Sol. Here, the polynomial $p(x)$ is

$$2x^3 + x^2 - 5x + 2$$

Value of the polynomial $2x^3 + x^2 - 5x + 2$

when $x = \frac{1}{2}$

$$= 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2 = \frac{1}{4} + \frac{1}{4} - \frac{5}{2} + 2 = 0$$

So, $\frac{1}{2}$ is a zero of $p(x)$.

On putting $x = 1$ in the cubic polynomial

$$2x^3 + x^2 - 5x + 2$$

$$= 2(1)^3 + (1)^2 - 5(1) + 2 = 2 + 1 - 5 + 2 = 0$$

On putting $x = -2$ in the cubic polynomial

$$2x^3 + x^2 - 5x + 2$$

$$= 2(-2)^3 + (-2)^2 - 5(-2) + 2$$

$$= -16 + 4 + 10 + 2 = 0$$

Hence, $\frac{1}{2}, 1, -2$ are the zeroes of the given polynomial.

Sum of the zeroes of $p(x)$

$$= \frac{1}{2} + 1 - 2 = -\frac{1}{2} = \frac{-\text{coefficient of } x^2}{\text{coefficient of } x^3}$$

Sum of the products of two zeroes taken at a time

$$= \frac{1}{2} \times 1 + \frac{1}{2} \times (-2) + 1 \times (-2)$$

$$= \frac{1}{2} - 1 - 2 = -\frac{5}{2} = \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$$

Product of all the three zeroes

$$= \left(\frac{1}{2}\right) \times (1) \times (-2) = -1 = \frac{-(-2)}{2} = \frac{-\text{constant term}}{\text{coefficient of } x^3}$$

Ex.109 Verify that the numbers given along sides of the cubic polynomial are their zeroes. Also verify relationship between the zeroes and the coefficient; $x^3 - 27x + 54$; $-6, 3, 3$

Sol. Here the polynomial $p(x)$ is $x^3 - 27x + 54$

Value of the polynomial $x^3 - 27x + 54$

when $x = -6$

$$= (-6)^3 - 27(-6) + 54 = -216 + 162 + 54 = 0$$

So, -6 is a zero of $p(x)$.

On putting $x = 3$ in the cubic polynomial

$$x^3 - 27x + 54 = (3)^3 - 27(3) + 54$$

$$= 27 - 81 + 54 = 0$$

So, 3 is a zero of $p(x)$.

Hence, $-6, 3, 3$ are the zeroes of the given cubic polynomial.

Sum of the zeroes of $p(x)$

$$= -6 + 3 + 3 = 0 = \frac{-(0)}{1} = \frac{-\text{coefficient of } x^2}{\text{coefficient of } x^3}$$

Sum of the product of two zeroes taken at a time

$$= (-6)(3) + (3)(3) + (-6)(3)$$

$$= -18 + 9 - 18 = -27 = \frac{-27}{1} = \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$$

Product of all the three zeroes

$$= (-6)(3)(3) = -54 = \frac{-54}{1} = \frac{-\text{constant term}}{\text{coefficient of } x^3}$$

Ex.110 Verify that the numbers given along side of the polynomial are their zeroes. Also verify the relationship between the zeroes and the coefficient.

$$x^4 + 2x^3 - 7x^2 - 8x + 12; -3, -2, 1, 2$$

Sol. Here the polynomial $p(x)$ is

$$x^4 + 2x^3 - 7x^2 - 8x + 12$$

Value of the polynomial $x^4 + 2x^3 - 7x^2 - 8x + 12$ when $x = -3$.

$$= (-3)^4 + 2(-3)^3 - 7(-3)^2 - 8(-3) + 12$$

$$= 81 - 54 - 63 + 24 + 12 = 0$$

So, -3 is a zero of $p(x)$.

On putting $x = -2$ in the given polynomial, we have

$$p(-2) = (-2)^4 + 2(-2)^3 - 7(-2)^2 - 8(-2) + 12$$

$$= 16 - 16 - 28 + 16 + 12 = 0$$

So, -2 is zero of $p(x)$.

On putting $x = 1$ in the given polynomial, we have

$$\begin{aligned} p(1) &= (1)^4 + 2(1)^3 - 7(1)^2 - 8(1) + 12 \\ &= 1 + 2 - 7 - 8 + 12 = 0 \end{aligned}$$

So, 1 is a zero of $p(x)$.

$$\begin{aligned} \text{On putting } x = 2 \text{ in the given polynomial, we have } p(2) &= (2)^4 + 2(2)^3 - 7(2)^2 - 8(2) + 12 \\ &= 16 + 16 - 28 - 16 + 12 = 0 \end{aligned}$$

So, 2 is zero of $p(x)$.

Hence, $-3, -2, 1, 2$ are the zeroes of given polynomial.

Sum of the zeroes of $p(x)$

$$= -3 - 2 + 1 + 2 = \frac{-2}{1} = \frac{-\text{coefficient of } x^3}{\text{coefficient of } x^4}$$

Sum of the products of two zeroes taken at a time

$$\begin{aligned} &= (-3)(-2) + (-2)(1) + (1)(2) + (-3)(1) \\ &\quad + (-3)(2) + (-2)(2) \end{aligned}$$

$$= 6 - 2 + 2 - 3 - 6 - 4 = -7 = \frac{-7}{1}$$

$$= \frac{\text{coefficient of } x^2}{\text{coefficient of } x^4}$$

Sum of the products of three zeroes taken at a time

$$\begin{aligned} &= (-3)(-2)(1) + (-2)(1)(2) \\ &\quad + (-3)(-2)(2) + (-3)(1)(2) \end{aligned}$$

$$= 6 - 4 + 12 - 6 = 8 = \frac{8}{1} = \frac{-(-8)}{1}$$

$$= -\frac{\text{coefficient of } x}{\text{coefficient of } x^4}$$

$$\text{Product of all the four zeroes} = (-3)(-2)(1)(2) = 12 = \frac{12}{1} = \frac{\text{constant term}}{\text{coefficient of } x^4}$$

Symmetric functions of Zeroes of a Quadratic Polynomial

◆ Symmetric function :

An algebraic expression in α and β , which remains unchanged, when α and β are interchanged is known as symmetric function in α and β .

For example, $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ etc. are symmetric functions. Symmetric function is to be expressed in terms of $(\alpha + \beta)$ and $\alpha\beta$. So, this can be evaluated for a given quadratic equation.

◆ Some useful relations involving α and β :

$$1. \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$2. (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$3. \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$4. \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$5. \alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$$

◆ EXAMPLES ◆

Ex.111 If α and β are the zeroes of the polynomial $ax^2 + bx + c$. Find the value of

(i) $\alpha - \beta$ (ii) $\alpha^2 + \beta^2$.

Sol. Since α and β are the zeroes of the polynomial $ax^2 + bx + c$.

$$\therefore \alpha + \beta = -\frac{b}{a}; \quad \alpha\beta = \frac{c}{a}$$

$$\begin{aligned} \text{(i)} \quad (\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= \left(-\frac{b}{a}\right)^2 - 4\frac{c}{a} = \frac{b^2}{a^2} - \frac{4ac}{a^2} = \frac{b^2 - 4ac}{a^2} \end{aligned}$$

$$\alpha - \beta = \frac{\sqrt{b^2 - 4ac}}{a}$$

$$\begin{aligned} \text{(ii)} \quad \alpha^2 + \beta^2 &= \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta = (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2 - 2ac}{a^2} \end{aligned}$$

Ex.112 If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$. Find the value of

(i) $\alpha^2 - \beta^2$ (ii) $\alpha^3 + \beta^3$.

Sol. Since α and β are the zeroes of $ax^2 + bx + c$

$$\therefore \alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

$$\text{(i)} \quad \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = -\frac{b}{a} \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$\begin{aligned} &= -\frac{b}{a} \sqrt{\left(-\frac{b}{a}\right)^2 - 4\frac{c}{a}} = -\frac{b}{a} \sqrt{\frac{b^2 - 4ac}{a^2}} \\ &= -\frac{b\sqrt{b^2 - 4ac}}{a^2} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \alpha^3 + \beta^3 &= (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) \\ &= (\alpha + \beta)[(\alpha^2 + \beta^2 + 2\alpha\beta) - 3\alpha\beta] \end{aligned}$$

$$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta] = \frac{-b}{a} \left[\left(-\frac{b}{a}\right)^2 - \frac{3c}{a} \right]$$

$$= \frac{-b}{a} \left[\frac{b^2}{a^2} - \frac{3c}{a} \right] = \frac{-b}{a} \left(\frac{b^2 - 3ac}{a^2} \right) = \frac{-b^3 + 3abc}{a^3}$$

To form a Quadratic Polynomial with the given Zeroes

Let zeroes of a quadratic polynomial be α and β .

$$\therefore x = \alpha, \quad x = \beta$$

$$x - \alpha = 0, \quad x - \beta = 0$$

The obviously the quadratic polynomial is

$$(x - \alpha)(x - \beta)$$

$$\text{i.e., } x^2 - (\alpha + \beta)x + \alpha\beta$$

$$x^2 - (\text{Sum of the zeroes})x + \text{Product of the zeroes}$$

◆ EXAMPLES ◆

Ex.113 Form the quadratic polynomial whose zeroes are 4 and 6.

Sol. Sum of the zeroes = $4 + 6 = 10$
 Product of the zeroes = $4 \times 6 = 24$
 Hence the polynomial formed
 = $x^2 - (\text{sum of zeroes})x + \text{Product of zeroes}$
 = $x^2 - 10x + 24$

Ex.114 Form the quadratic polynomial whose zeroes are -3, 5.

Sol. Here, zeroes are -3 and 5.
 Sum of the zeroes = $-3 + 5 = 2$
 Product of the zeroes = $(-3) \times 5 = -15$
 Hence the polynomial formed
 = $x^2 - (\text{sum of zeroes})x + \text{Product of zeroes}$
 = $x^2 - 2x - 15$

Ex.115 Find a quadratic polynomial whose sum of zeroes and product of zeroes are respectively -

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

Sol. Let the polynomial be $ax^2 + bx + c$ and its zeroes be α and β .

(i) Here, $\alpha + \beta = \frac{1}{4}$ and $\alpha \cdot \beta = -1$

Thus the polynomial formed
 = $x^2 - (\text{Sum of zeroes})x + \text{Product of zeroes}$
 = $x^2 - \left(\frac{1}{4}\right)x - 1 = x^2 - \frac{x}{4} - 1$

The other polynomial are $k\left(x^2 - \frac{x}{4} - 1\right)$

If $k = 4$, then the polynomial is $4x^2 - x - 4$.

(ii) Here, $\alpha + \beta = \sqrt{2}$ $\alpha\beta = \frac{1}{3}$

Thus the polynomial formed
 = $x^2 - (\text{Sum of zeroes})x + \text{Product of zeroes}$
 = $x^2 - (\sqrt{2})x + \frac{1}{3}$ or $x^2 - \sqrt{2}x + \frac{1}{3}$

Other polynomial are $k\left(x^2 - \sqrt{2}x + \frac{1}{3}\right)$

If $k = 3$, then the polynomial is $3x^2 - 3\sqrt{2}x + 1$

(iii) Here, $\alpha + \beta = 0$ and $\alpha \cdot \beta = \sqrt{5}$

Thus the polynomial formed

$$= x^2 - (\text{Sum of zeroes}) x + \text{Product of zeroes}$$

$$= x^2 - (0) x + \sqrt{5} = x^2 + \sqrt{5}$$

Ex.116 Find a cubic polynomial with the sum of its zeroes, sum of the products of its zeroes taken two at a time, and product of its zeroes as 2, -7 and -14, respectively.

Sol. Let the cubic polynomial be

$$ax^3 + bx^2 + cx + d$$

$$\Rightarrow x^3 + \frac{b}{a} x^2 + \frac{c}{a} x + \frac{d}{a} \quad \dots(1)$$

and its zeroes are α , β and γ , then

$$\alpha + \beta + \gamma = 2 = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = -7 = \frac{c}{a}$$

$$\alpha\beta\gamma = -14 = -\frac{d}{a}$$

Putting the values of $\frac{b}{a}$, $\frac{c}{a}$ and $\frac{d}{a}$ in (1),

we get

$$x^3 + (-2) x^2 + (-7)x + 14$$

$$\Rightarrow x^3 - 2x^2 - 7x + 14$$

Ex.117 Find the cubic polynomial with the sum, sum of the product of its zeroes taken two at a time and product of its zeroes as 0, -7 and -6 respectively.

Sol. Let the cubic polynomial be

$$ax^3 + bx^2 + cx + d$$

$$\Rightarrow x^3 + \frac{b}{a} x^2 + \frac{c}{a} x + \frac{d}{a} \quad \dots(1)$$

and its zeroes are α , β , γ . Then

$$\alpha + \beta + \gamma = 0 = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = -7 = \frac{c}{a}$$

$$\alpha\beta\gamma = -6 = \frac{-d}{a}$$

Putting the values of $\frac{b}{a}$, $\frac{c}{a}$ and $\frac{d}{a}$ in (1),

we get

$$x^3 - (0) x^2 + (-7) (-6)$$

$$\text{or } x^3 - 7x + 6$$

Ex.118 If α and β are the zeroes of the polynomials $ax^2 + bx + c$ then form the polynomial whose zeroes are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Sol. Since α and β are the zeroes of $ax^2 + bx + c$

$$\alpha + \beta = \frac{-b}{a}, \quad \alpha\beta = \frac{c}{a}$$

$$\text{Sum of the zeroes} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$$

$$= \frac{\frac{-b}{a}}{\frac{c}{a}} = -\frac{b}{c}$$

Product of the zeroes

$$= \frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{a}{c}$$

But required polynomial is

$$x^2 - (\text{sum of zeroes})x + \text{Product of zeroes}$$

$$\Rightarrow x^2 - \left(\frac{-b}{c}\right)x + \left(\frac{a}{c}\right)$$

$$\text{or } x^2 + \frac{b}{c}x + \frac{a}{c} \text{ or } c\left(x^2 + \frac{b}{c}x + \frac{a}{c}\right)$$

$$\Rightarrow cx^2 + bx + a$$

Ex.119 If a and b are the zeroes of the polynomial

$$x^2 + 4x + 3, \text{ form the polynomial whose zeroes are } 1 + \frac{\beta}{\alpha} \text{ and } 1 + \frac{\alpha}{\beta}.$$

Sol. Since α and β are the zeroes of the polynomial

$$x^2 + 4x + 3 \text{ Then, } \alpha + \beta = -4, \quad \alpha\beta = 3$$

Sum of the zeroes

$$= 1 + \frac{\beta}{\alpha} + 1 + \frac{\alpha}{\beta} = \frac{\alpha\beta + \beta^2 + \alpha\beta + \alpha^2}{\alpha\beta}$$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta}{\alpha\beta} = \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{(-4)^2}{3} = \frac{16}{3}$$

Product of the zeroes

$$= \left(1 + \frac{\beta}{\alpha}\right)\left(1 + \frac{\alpha}{\beta}\right) = 1 + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{\alpha\beta}$$

$$= 2 + \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{2\alpha\beta + \alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{(-4)^2}{3} = \frac{16}{3}$$

But required polynomial is

$$x^2 - (\text{sum of zeroes})x + \text{product of zeroes}$$

$$\text{or } x^2 - \frac{16}{3}x + \frac{16}{3} \text{ or } k\left(x^2 - \frac{16}{3}x + \frac{16}{3}\right)$$

$$\text{or } 3\left(x^2 - \frac{16}{3}x + \frac{16}{3}\right) \text{ (if } k = 3)$$

$$\Rightarrow 3x^2 - 16x + 16$$

Working rule to Divide a Polynomial by Another Polynomial

Step 1: First arrange the term of dividend and the divisor in the decreasing order of their degrees.

Step 2: To obtain the first term of quotient divide the highest degree term of the dividend by the highest degree term of the divisor.

Step 3: To obtain the second term of the quotient, divide the highest degree term of the new dividend obtained as remainder by the highest degree term of the divisor.

Step 4: Continue this process till the degree of remainder is less than the degree of divisor.

◆ Division Algorithm for Polynomial

If $p(x)$ and $g(x)$ are any two polynomials with $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that

$$p(x) = q(x) \times g(x) + r(x)$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

The result is called Division Algorithm for polynomials.

$$\boxed{\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}}$$

◆ EXAMPLES ◆

Ex.120 Divide $3x^3 + 16x^2 + 21x + 20$ by $x + 4$.

Sol.

$$\begin{array}{r}
 3x^2 + 4x + 5 \\
 x + 4 \overline{) 3x^3 + 16x^2 + 21x + 20} \\
 \underline{3x^3 + 12x^2} \\
 4x^2 + 21x + 20 \\
 \underline{4x^2 + 16x} \\
 5x + 20 \\
 \underline{5x + 20} \\
 0
 \end{array}$$

First term of $q(x) = \frac{3x^3}{x} = 3x^2$

Second term of $q(x) = \frac{4x^2}{x} = 4x$

Third term of $q(x) = \frac{5x}{x} = 5$

$$\text{Quotient} = 3x^2 + 4x + 5$$

$$\text{Remainder} = 0$$

Ex.121 Apply the division algorithm to find the quotient and remainder on dividing $p(x)$ by $g(x)$ as given below :

$$p(x) = x^3 - 3x^2 + 5x - 3, \quad g(x) = x^2 - 2$$

Sol. We have,

$$p(x) = x^3 - 3x^2 + 5x - 3 \text{ and } g(x) = x^2 - 2$$

$$\begin{array}{r}
 x^2 - 2 \overline{) \begin{array}{l} x^3 - 3x^2 + 5x - 3 \\ x^3 - 2x \\ \hline -3x^2 + 7x - 3 \\ -3x^2 + 6 \\ \hline 7x - 9 \end{array} } \\
 \text{First term of quotient is } \frac{x^3}{x^2} = x \\
 \text{Second term of quotient is } \frac{-3x^2}{x^2} = -3
 \end{array}$$

We stop here since

So, quotient = $x - 3$, remainder = $7x - 9$

Therefore,

$$\begin{aligned}
 &\text{Quotient} \times \text{Divisor} + \text{Remainder} \\
 &= (x - 3)(x^2 - 2) + 7x - 9 \\
 &= x^3 - 2x - 3x^2 + 6 + 7x - 9 \\
 &= x^3 - 3x^2 + 5x - 3 = \text{Dividend}
 \end{aligned}$$

Therefore, the division algorithm is verified.

Ex.122 Apply the division algorithm to find the quotient and remainder on dividing $p(x)$ by $g(x)$ as given below

$$p(x) = x^4 - 3x^2 + 4x + 5, \quad g(x) = x^2 + 1 - x$$

Sol. We have,

$$p(x) = x^4 - 3x^2 + 4x + 5, \quad g(x) = x^2 + 1 - x$$

$$\begin{array}{r}
 x^2 - x + 1 \overline{) \begin{array}{l} x^4 - 3x^2 + 4x + 5 \\ x^4 - x^3 + x^2 \\ \hline x^3 - 4x^2 + 4x + 5 \\ x^3 - x^2 + x \\ \hline -3x^2 + 3x + 5 \\ -3x^2 + 3x - 3 \\ \hline 8 \end{array} }
 \end{array}$$

We stop here since

degree of (8) < degree of $(x^2 - x + 1)$.

So, quotient = $x^2 + x - 3$, remainder = 8

Therefore,

$$\begin{aligned}
 &\text{Quotient} \times \text{Divisor} + \text{Remainder} \\
 &= (x^2 + x - 3)(x^2 - x + 1) + 8 \\
 &= x^4 - x^3 + x^2 + x^3 - x^2 + x - 3x^2 + 3x - 3 + 8 \\
 &= x^4 - 3x^2 + 4x + 5 = \text{Dividend}
 \end{aligned}$$

Therefore the Division Algorithm is verified.

Ex.123 Check whether the first polynomial is a factor of the second polynomial by applying the division algorithm. $t^2 - 3$; $2t^4 + 3t^3 - 2t^2 - 9t - 12$.

Sol. We divide $2t^4 + 3t^3 - 2t^2 - 9t - 12$ by $t^2 - 3$

$$\begin{array}{r}
 2t^2 + 3t + 4 \\
 t^2 - 3 \overline{) 2t^4 + 3t^3 - 2t^2 - 9t - 12} \\
 \underline{2t^4 - 6t^2} \\
 3t^3 + 4t^2 + 9t - 12 \\
 \underline{3t^3 - 9t} \\
 4t^2 - 12 \\
 \underline{4t^2 - 12} \\
 0
 \end{array}$$

Here, remainder is 0, so $t^2 - 3$ is a factor of $2t^4 + 3t^3 - 2t^2 - 9t - 12$.

$$2t^4 + 3t^3 - 2t^2 - 9t - 12 = (2t^2 + 3t + 4)(t^2 - 3)$$

Ex.124 Check whether first polynomial is a factor of the second polynomial by applying the division algorithm.

$$x^2 + 3x + 1, 3x^4 + 5x^3 - 7x^2 + 2x + 2$$

Sol. We divide $3x^4 + 5x^3 - 7x^2 + 2x + 2$ by $x^2 + 3x + 1$

$$\begin{array}{r}
 3x^2 + 4x + 2 \\
 x^2 - 3x + 1 \overline{) 3x^4 + 5x^3 - 7x^2 + 2x + 2} \\
 \underline{3x^4 + 9x^3 + 3x^2} \\
 -4x^3 - 10x^2 + 2x + 2 \\
 \underline{-4x^3 - 12x^2 - 4x} \\
 2x^2 + 6x + 2 \\
 \underline{2x^2 + 6x^2 + 2} \\
 0
 \end{array}$$

Since, here remainder is zero.

Hence, $x^2 + 3x + 1$ is a factor of $3x^4 + 5x^3 - 7x^2 + 2x + 2$.

Checking

$$\begin{aligned}
 &3x^4 + 5x^3 - 7x^2 + 2x + 2 \\
 &= (3x^2 - 4x + 2)(x^2 + 3x + 1) + 0 \\
 &= 3x^4 + 5x^3 - 7x^2 + 2x + 2 = \text{Dividend}
 \end{aligned}$$

Ex.125 Check whether the first polynomial is a factor of the second polynomial by applying the division algorithm.

$$x^3 - 3x + 1, x^5 - 4x^3 + x^2 + 3x + 1$$

Sol. We divide $x^5 - 4x^3 + x^2 + 3x + 1$ by $x^3 - 3x + 1$.

$$\begin{array}{r}
 x^3 - 3x + 1 \overline{) x^5 - 4x^3 + x^2 + 3x + 1} \\
 \underline{x^5 - 3x^3 + x^2} \\
 -x^3 + 3x + 1 \\
 \underline{-x^3 + 3x - 1} \\
 3x - 1 + 1 \\
 3x \\
 \underline{3x - 1} \\
 0
 \end{array}$$

Here, remainder is 2, so $x^3 - 3x + 1$ is not a factor of $x^5 - 4x^3 + x^2 + 3x + 1$.

Checking :

$$\begin{aligned}
 x^5 - 4x^3 + x^2 + 3x + 1 &= (x^2 - 1)(x^3 - 3x + 1) + 2 \\
 &= (x^5 - 4x^3 + x^2 + 3x - 1) + 2
 \end{aligned}$$

Ex.126 Obtain all the zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

Sol. Since two zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$,

$$x = \sqrt{\frac{5}{3}}, x = -\sqrt{\frac{5}{3}}$$

$$\Rightarrow \left(x - \sqrt{\frac{5}{3}}\right)\left(x + \sqrt{\frac{5}{3}}\right) = x^2 - \frac{5}{3} \text{ or } 3x^2 - 5 \text{ is a factor of the given polynomial.}$$

Now, we apply the division algorithm to the given polynomial and $3x^2 - 5$.

$$\begin{array}{r}
 3x^2 - 5 \overline{) 3x^4 + 6x^3 - 2x^2 - 10x - 5} \\
 \underline{3x^4 - 5x^2} \\
 6x^3 + 3x^2 - 10x - 5 \\
 \underline{6x^3 - 10x} \\
 3x^2 - 5 \\
 \underline{3x^2 - 5} \\
 0
 \end{array}$$

$$\begin{aligned}
 \text{So, } 3x^4 + 6x^3 - 2x^2 - 10x - 5 \\
 = (3x^2 - 5)(x^2 + 2x + 1) + 0
 \end{aligned}$$

$$\text{Quotient} = x^2 + 2x + 1 = (x + 1)^2$$

Zeroes of $(x + 1)^2$ are $-1, -1$.

Hence, all its zeroes are $\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}, -1, -1$

Ex.127 On dividing $x^3 - 3x^2 + x + 2$ by a polynomial $g(x)$, the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find $g(x)$.

Sol. $p(x) = x^3 - 3x^2 + x + 2$

$$q(x) = x - 2 \text{ and } r(x) = -2x + 4$$

By Division Algorithm, we know that

$$p(x) = q(x) \times g(x) + r(x)$$

Therefore,

$$x^3 - 3x^2 + x + 2 = (x - 2) \times g(x) + (-2x + 4)$$

$$\Rightarrow x^3 - 3x^2 + x + 2 + 2x - 4 = (x - 2) \times g(x)$$

$$\Rightarrow g(x) = \frac{x^3 - 3x^2 + 3x - 2}{x - 2}$$

On dividing $x^3 - 3x^2 + 3x - 2$ by $x - 2$, we get $g(x)$

$$\begin{array}{r}
 x-2 \overline{) \begin{array}{r} x^3 - 3x^2 + 3x - 2 \\ x^3 - 2x^2 \\ \hline -x^2 + 3x - 2 \\ -x^2 + 2x \\ \hline x - 2 \\ x - 2 \\ \hline 0 \end{array} \\
 \end{array}$$

First term of quotient is $\frac{x^3}{x} = x$

Second term of quotient is $\frac{-x^2}{x} = -x$

Third term of quotient is $\frac{x}{x} = 1$

Hence, $g(x) = x^2 - x + 1$.

Ex.128 Give examples of polynomials $p(x)$, $q(x)$ and $r(x)$, which satisfy the division algorithm and

(i) $\deg p(x) = \deg q(x)$

(ii) $\deg q(x) = \deg r(x)$

(iii) $\deg q(x) = 0$

Sol. (i) Let $q(x) = 3x^2 + 2x + 6$, degree of $q(x) = 2$

$p(x) = 12x^2 + 8x + 24$, degree of $p(x) = 2$

Here, $\deg p(x) = \deg q(x)$

(ii) $p(x) = x^5 + 2x^4 + 3x^3 + 5x^2 + 2$

$q(x) = x^2 + x + 1$, degree of $q(x) = 2$

$g(x) = x^3 + x^2 + x + 1$

$r(x) = 2x^2 - 2x + 1$, degree of $r(x) = 2$

Here, $\deg q(x) = \deg r(x)$

(iii) Let $p(x) = 2x^4 + 8x^3 + 6x^2 + 4x + 12$

$q(x) = 2$, degree of $q(x) = 0$

$g(x) = x^4 + 4x^3 + 3x^2 + 2x + 6$

$r(x) = 0$

Here, $\deg q(x) = 0$

EXERCISE # 1

VERY SHORT ANSWER TYPE QUESTIONS

Factorize each of the following expression

Q.1 $x^2 - x - 42$

Q.2 $6 - 5y - y^2$

Q.3 $a^2 + 46a + 205$

Q.4 $ab + ac - b^2 - bc$

Q.5 $p^4 - 81q^4$

Use remainder theorem to find remainder, when $p(x)$ is divided by $q(x)$ in following questions.

Q.6 $p(x) = 2x^2 - 5x + 7, q(x) = x - 1$

Q.7 $p(x) = x^9 - 5x^4 + 1, q(x) = x + 1$

Q.8 $p(x) = 2x^3 - 3x^2 + 4x - 1, q(x) = x + 2$

Use factor theorem to verify in each of the following that $q(x)$ is a factor of $p(x)$.

Q.9 $p(x) = 2x^3 + 5x^2 + 3x - 4, q(x) = x - 1$

Q.10 $p(x) = x^4 - 3x^2 + 2x + 1, q(x) = x - 1$

SHORT ANSWER TYPE QUESTIONS

Factorize each of the following expression

Q.11 $125a^3 + \frac{b^3}{27}$

Q.12 If one of the factors of $x^2 + x - 20$ is $(x + 5)$, find other factor.

Q.13 Find positive square root of $36x^2 + 60x + 25$

Q.14 Simplify : $\sqrt{2a^2 + 2\sqrt{6ab} + 3b^2}$

Q.15 $(x^2 + 4y)^2 + 21(x^2 + 4y) + 98$

Q.16 $4(x-y)^2 - 12(x-y)(x+y) + 9(x+y)^2$

Use factor theorem to verify in each of the following that $q(x)$ is a factor of $p(x)$.

Q.17 Find the value of k if $(x - 2)$ is a factor of $2x^3 - 6x^2 + 5x + k$

Q.18 Find the value of k if $(x+3)$ is a factor of $3x^2 + kx + 6$.

Q.19 $p(x) = 3x^6 - 7x^5 + 7x^4 - 3x^3 + 2x^2 - 2, q(x) = x - 1$

Q.20 For what value of k is $y^3 + ky + 2k - 2$ exactly divisible by $(y + 1)$?

LONG ANSWER TYPE QUESTIONS

Q.21 Show that if $a + b$ is not zero, then the equation $a(x-a) = 2ab - b(x-b)$ has a solution $x = a + b$.

Q.22 Prove that

$$a^3 + b^3 + c^3 - 3abc$$

$$= \frac{1}{2}(a+b+c) [(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Q.23 Prove that

$$(a+b)^3 + (b+c)^3 + (c+a)^3 - 3(a+b)(b+c)(c+a) = 2(a^3 + b^3 + c^3 - 3abc)$$

Q.24 If $x+1$ and $x-1$ are factors of

$$mx^3 + x^2 - 2x + n, \text{ find the value of } m \text{ and } n.$$

Q.25 If $x^3 + mx^2 + nx + 6$ has $x-2$ as a factor and leaves a remainder 3, when divided by $x-3$, find use of m and n .

Q.26 What must be subtracted from $4x^4 - 2x^3 - 6x^2 + x - 5$ so that the result is exactly divisible by $2x^2 + x - 1$

Q.27 Using factor theorem, factorize the polynomial $x^4 + x^3 - 7x^2 - x + 6$.

Q.28 Let A and B are the remainders when the polynomial $y^3 + 2y^2 - 5ay - 7$ and $y^3 + ay^2 - 12y + 6$ are divided by $y+1$ and $y-2$ respectively. If $2A + B = 6$, find the value of a .

Q.29 If $(3x-1)^4 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, then find the value of $a_4 + 3a_3 + 9a_2 + 27a_1 + 81a_0$.

FILL IN THE BLANKS

Q.30 If $a + b = 2$, then the value of $a^3 + b^3 + 6ab$ is equal to

Q.31 The coefficient of 'a' in $(a + b + c)(x + y + z) + (b + c - a)(x + y - z) + (c + a - b)(y + z - x)$ is

Q.32 $\{(x+y)^2 + (x-y)^2\} + \left[\frac{x}{y} + \frac{y}{x}\right]$ reduces to

Q.33 The expression

$$\frac{A}{3x-1} + \frac{B}{x+4} = \frac{9-x}{3x^2+11x-4} \text{ for all values of } x. \text{ Then } A \text{ and } B \text{ are respectively equal to } \dots\dots\dots$$

Q.34 The coefficient of x^2 in $(px^2 + 4x + r) \times (4x^2 - 3qx - 5)$ is

Q.35 The value of

$$\{x+(y-z)\}^2 - \{y-(z+x)\}^2 + \{z-(x+y)\}^2$$

when $x = 1, y = -2, c = 3$ is

Q.36 $\frac{5m^2 - 4m - 2}{5m^2 - 4m} = \frac{7}{2}$ then the value of m

Q.37 $abx^2 + (a^2 - b^2)x - ab$ when expressed in linear factors is equal to

Q.38 $x^2(y - z) + y^2(z - x) + z^2(x - y) = N(x - y)(y - z)(z - x)$, then N must have the value

Q.39 $4x^2 - 2bx + ab - a^2$ when expressed in linear factors is equal to

TRUE/FALSE TYPE QUESTIONS

- Q.40** When $14x^3 - 3x^2 + 4x + 2$ is divided by $2x-1$, the remainder is 5.
- Q.41** $4x^2 - 8x + 15$ is exactly divisible by $2x-1$.
- Q.42** $x^2 - 5x + 6$ cannot be written as a product of two linear factors.
- Q.43** $x^3 - 3x^2y + 3xy^2 - y^3$ is exactly divisible by $x - y$.
- Q.44** $(2a + b)^2 - (2b + a)^2 = 3(a^2 - b^2)$ is an identity
- Q.45** $9 + 4\sqrt{5}$ and $9 - 4\sqrt{5}$ are reciprocal numbers.
- Q.46** $(x^2 + 1)^4 = (x^4 - 6x^2 + 1)^2 + 16(x^3 - x)^2$ is an identity.
- Q.47** $3x + [4x - \{3x + (4x - 5) + 3\} - 5x]$ is independent of x .

SINGLE CHOICE TYPE QUESTIONS

- Q.48** Take away the sum of squares of a and b from the square of the sum of a and b , and divide the number obtained by ab . Then the resulting number is -
 (A) a (B) b (C) 2 (D) None
- Q.49** If x be a natural number, then the number of possible solutions of the equation $ax = x + 6$ for various values of a is
 (A) 1 (B) 2 (C) 4 (D) None
- Q.50** If $x + \frac{1}{x} = 2$ then the value of $x^2 + \frac{1}{x^2}$ is
 (A) 2 (B) 8 (C) $31\frac{1}{2}$ (D) None
- Q.51** If $a + b + c + d = 0$ then $(a + d)^2 + (b + d)^2 + (c + d)^2$ is equal to
 (A) $a^2 + b^2 + c^2 + d^2$ (B) $a^2 + b^2 + c^2 + 3d^2$
 (C) $a^2 + 2b^2 + 3c^2 + 4d^2$ (D) None
- Q.52** An expression when divided by $x + 1$ has $x^2 - x + 1$ for quotient and 3 for the remainder. It is equal to
 (A) $x^3 + 4$ (B) $x^3 - 2$ (C) $x^3 - 1$ (D) None
- Q.53** If $a = x^3 + 3x^2 + 3x + 1$; $b = x^2 + 2x + 1$;
 $c = x + 1$, then the value of $a - 3b + 3c - 1$ when $x = -\frac{1}{3}$ is
 (A) $\frac{1}{9}$ (B) $-\frac{1}{27}$ (C) $-\frac{1}{36}$ (D) None
- Q.54** If $x + \frac{a}{x} = y + \frac{a}{y}$ then
 (A) $x = ay$ (B) $y = ax$ (C) $xy = a$ (D) None
- Q.55** If $a + b - c = 0$, then
 (A) $b^2 + ac = c^2 - ab$ (B) $c^2 + ab = a^2 - bc$
 (C) $a^2 + bc = b^2 - ca$ (D) None

- Q.56** If $a + b + c = 0$, then the value of $\frac{a^2}{a^2 - bc} + \frac{b^2}{b^2 - ca} + \frac{c^2}{c^2 - ab}$ is
 (A) 1 (B) 2 (C) 3 (D) None
- Q.57** Let $p = \frac{1-m^2}{(1+mn)^2 - (m+n)^2}$, then $p(1-n^2)$ is equal to -
 (A) m^2 (B) n^2 (C) 1 (D) None

ONE OR MORE THAN ONE CORRECT CHOICE TYPE QUESTIONS

- Q.58** If $x - 1$ is a factor of $k^2x^3 - 4kx + 4k - 1$, then the value of k is -
 (A) 1 (B) -1 (C) 2 (D) -2
- Q.59** For what value of a is the polynomial $2x^4 - ax^3 + 4x^2 + 2x + 1$ divisible by $1 - 2x$?
 (A) $a = 25$ (B) $a = 24$ (C) $a = 23$ (D) $a = 22$
- Q.60** If $x + 1$ is a factor of polynomial $2x^3 - 2x^2 + x + p$, then which of these are not the value of 'p' -
 (A) 2 (B) 3 (C) 5 (D) 4
- Q.61** If $x + a$ is a factor of $x^3 + ax^2 - 2x + a$ then which of these are not the values of a
 (A) $4/3$ (B) $-4/3$ (C) $5/3$ (D) $-5/3$
- Q.62** If one of the factors of $x^2 + x - 20$ is $(x+5)$, then other factor is -
 (A) $(x - 4)$ (B) $(x - 5)$
 (C) $(x - 6)$ (D) $(x - 7)$
- Q.63** Which of these are not the positive square root of $36x^2 + 60x + 25$
 (A) $6x - 4$ (B) $6x + 5$ (C) $6x + 3$ (D) $6x - 3$

MATCH THE COLUMN TYPE QUESTIONS

Q.64. Match the column

Column 1

Column 2

- (i) If $a + b$ is not zero, then the equation $a(x-a) = 2ab - b(x-b)$ has a solution
- (ii) If $2(a^2 + b^2) = (a+b)^2$, then
- (A) $a = b$
 (B) $x = a + b$
 (C) $x = a - b$

Q.65. Match the column

Column 1

Column 2

- (i) If a polynomial $p(x)$ is divided by $(x+a)$, the remainder is the value of $p(x)$, at
- (ii) If a polynomial $p(x)$ is divided by $(ax-b)$, the
- (A) $x = -a$
 (B) $x = b/a$

remainder is the value
of $p(x)$ at

Q.66 Match the column

Column 1

Column 2

(i) Factorization of

(A) $\frac{1}{3}(x-9)(x+3)$

$\frac{1}{3}x^2 - 2x - 9$ is

(B) $\frac{1}{3}(x+9)(x+3)$

(ii) Factorization of

$100 - 9p^2$ is

(C) $(10 - 3p)(10 + 3p)$

(D) $(100 - 3p)(100 + 3p)$

Q.67 Match the column

Column 1

Column 2

(i) Factors of

(A) $(x+1)(x-1)(x-3)$

$2x^4 + x^3 - 14x^2 - 19x - 6$ are

(ii) Factors of

(B) $(x+1)(x+2)(x-3)$

$x^3 - 3x^2 - x + 3$ are

$(2x+1)$

Assertion & Reasoning Type Questions

The following questions consists of two statements each, printed as Assertion and Reason. While answering these questions you are to choose any one of the following four responses.

(A) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion.

(B) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion.

(C) If Assertion is true but the Reason is false.

(D) If Assertion is false but Reason is true.

Q.68 **Assertion :** The remainder when the polynomial $p(x) = x^4 - 3x^2 + 2x + 1$ is divided by $x-1$ is $p(1) = 1$

Reason : $p(x) = x^4 - 3x^2 + 2x + 1$

$$p(1) = 1^4 - 3 \times 1^2 + 2 \times 1 + 1$$

$$= 1 - 3 + 2 + 1 = 1$$

Hence, required remainder = $p(1) = 1$

Q.69 **Assertion :** $(x - 3)$ is a factor of the polynomial $x^3 - 3x^2 + 4x - 12$

Reason : $(x - 1)$ is a factor of $x^{10} - 1$ and also of $x^{11} - 1$

Q.70 **Assertion :** $(x + a)$ is a factor of $x^n + a^n$ for any odd positive integer.

Reason : In order to prove that $x + a$ is a factor of $f(x)$ for any odd positive integer n , it is sufficient to show that $f(a) = 0$

Q.71 **Assertion :** $(ax - b)$ is a factor of polynomial $p(x)$ if $p\left(\frac{a}{b}\right) = 0$

Reason : $(x - a)(x - b)$ is a factor of polynomial $p(x)$ if $p(a) = 0$ and $p(b) = 0$

Q.72 Assertion : If both $x - 2$ and $x - 1/2$ are factors of $px^2 + 5x + r$, then $p = r$

Reason : If $x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$ then $a + c + e = b + d = 0$

ANSWER KEY

A. VERY SHORT ANSWER TYPE :

1. $(x + 6)(x - 7)$ 2. $(6 + y)(1 - y)$
 3. $(a + 41)(a + 5)$ 4. $(a - b)(b + c)$
 5. $(p + 3q)(p - 3q)(p^2 + 9q^2)$
 6. 4 7. -5 8. -37

B. SHORT ANSWER TYPE :

11. $\left(5a + \frac{b}{3}\right) \left[25a^2 - \frac{5ab}{3} + \frac{b^2}{9}\right]$
 12. $(x - 4)$ 13. $6x + 5$ 14. $(\sqrt{2}a + \sqrt{3}b)$
 15. $(x^2 + 4y + 7)(x^2 + 4y + 14)$ 16. $(x + 5y)^2$
 17. -2 18. 11 20. 3

C. LONG ANSWER TYPE :

24. $m = 2, n = -1$ 25. $m = -3, n = -1$
 26. -6 27. $(x - 1)(x + 1)(x - 2)(x + 3)$
 28. $a = x$ 29. 0

D. FILL IN THE BLANKS :

30. 8 31. $-x + y + 3z$ 32. $2xy$
 33. $A = 2, B = -1$ 34. $-5p - 12q + 4r$ 35. -4
 36. $2/5$ 37. $(bx + a)(ax - b)$
 38. $N = -1$ 39. $(2x - a)(2x - b + a)$

E. TRUE/FALSE TYPE QUESTIONS :

40. True 41. False 42. False 43. True
 44. True 45. True 46. True 47. False

F. SINGLE CHOICE TYPE QUESTIONS :

48. (C) 49. (C) 50. (A) 51. (A)
 52. (A) 53. (B) 54. (C) 55. (A)
 56. (B) 57. (C)

G. ONE OR MORE THAN ONE CORRECT CHOICE TYPE QUESTIONS :

58. (A), (B) 59. (A)
 60. (A), (B), (D) 61. (A), (C), (D)
 62. (A) 63. (A), (C), (D)

H. MATCH THE COLUMN TYPE QUESTIONS :

64. (i) - (B), (ii) - (A)
 65. (i) - (A), (ii) - (B)
 66. (i) - (A), (ii) - (C)
 67. (i) - (B), (ii) - (A)

I. ASSERTION AND REASON TYPE QUESTIONS :

68. (A) 69. (B) 70. (C)
 71. (D) 72. (B)

EXERCISE # 2

Single Choice Questions

- Q.1** If $\left(x + \frac{1}{x}\right) = 3$, then $\left(x^2 + \frac{1}{x^2}\right)$ is equal to -
(A) $\frac{10}{3}$ (B) $\frac{82}{9}$ (C) 7 (D) 11
- Q.2** If $\left(x - \frac{1}{x}\right) = \frac{1}{2}$, then $\left(4x^2 + \frac{4}{x^2}\right)$ is equal to -
(A) 7 (B) -7 (C) 9 (D) -9
- Q.3** If $\left(x + \frac{1}{x}\right) = 4$, then $\left(x^4 + \frac{1}{x^4}\right)$ is equal to -
(A) 196 (B) 194 (C) 192 (D) 190
- Q.4** If $\left(x + \frac{1}{x}\right) = \sqrt{3}$, then the value of $\left(x^3 + \frac{1}{x^3}\right)$ is -
(A) 0 (B) $3\sqrt{3}$
(C) $3(\sqrt{3} - 1)$ (D) $3(\sqrt{3} + 1)$
- Q.5** If $\left(x + \frac{1}{x}\right) = 2$, then $\left(x^3 + \frac{1}{x^3}\right)$ is equal to -
(A) 64 (B) 14 (C) 8 (D) 2
- Q.6** If $\left(x^2 + \frac{1}{x^2}\right) = 102$, the value of $\left(x - \frac{1}{x}\right)$ is -
(A) 8 (B) 10 (C) 12 (D) 13
- Q.7** If $\left(x^4 + \frac{1}{x^4}\right) = 322$, the value of $\left(x - \frac{1}{x}\right)$ is -
(A) 4 (B) 6 (C) 8 (D) $3\sqrt{2}$
- Q.8** If $\left(x^3 + \frac{1}{x^3}\right) = 52$, the value of $\left(x + \frac{1}{x}\right)$ is -
(A) 4 (B) 3 (C) 6 (D) 13
- Q.9** If $\left(x^3 - \frac{1}{x^3}\right) = 14$, the value of $\left(x - \frac{1}{x}\right)$ is -
(A) 5 (B) 4 (C) 3 (D) 2
- Q.10** If x is an integer such that $\left(x + \frac{1}{x}\right) = \left(\frac{17}{4}\right)$, then the value of $\left(x - \frac{1}{x}\right)$ is -
(A) 4 (B) $\frac{13}{4}$ (C) $\frac{15}{4}$ (D) $\frac{1}{4}$
- Q.11** If $\left(x^4 + \frac{1}{x^4}\right) = 727$, the value of $\left(x^3 - \frac{1}{x^3}\right)$ is
(A) 125 (B) 140 (C) 155 (D) 170
- Q.12** If $\left(2x - \frac{3}{x}\right) = 5$, the value of $\left(4x^2 - \frac{9}{x^2}\right)$ is -
(A) 25 (B) 30 (C) 35 (D) 49

- Q.13** If $\left(x + \frac{1}{x}\right) = 3$, the value of $\left(x^6 + \frac{1}{x^6}\right)$ is -
 (A) 927 (B) 414 (C) 364 (D) 322
- Q.14** If $t^2 - 4t + 1 = 0$, then the value of $\left(t^3 + \frac{1}{t^3}\right)$ is -
 (A) 44 (B) 48 (C) 52 (D) 64
- Q.15** If $x + y = 7$ and $xy = 12$, the value of $(x^2 + y^2)$ is -
 (A) 25 (B) 29 (C) 37 (D) 49
- Q.16** If $x + y = 5$ and $xy = 6$, the value of $(x^3 + y^3)$ is -
 (A) 91 (B) 133 (C) 217 (D) 343
- Q.17** If $x + y = 5$ and $xy = 6$, the value of $(x^3 - y^3)$ is -
 (A) -19 (B) 19 (C) -63 (D) 63
- Q.18** If $x^{1/3} + y^{1/3} + z^{1/3} = 0$, then -
 (A) $x + y + z = 0$ (B) $(x + y + z)^3 = 27xyz$
 (C) $x + y + z = 3xyz$ (D) $x^3 + y^3 + z^3 = 0$
- Q.19** If $a + b + c = 0$, then $(a^3 + b^3 + c^3)$ is equal to -
 (A) 0 (B) abc
 (C) $3abc$ (D) $(ab + bc + ca)$
- Q.20** If $a + b + c = 0$, then the value of $\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right)$ is -
 (A) 1 (B) 0 (C) -1 (D) 3
- Q.21** If $x + y + z = 9$ & $xy + yz + zx = 23$, then the value of $(x^3 + y^3 + z^3 - 3xyz)$ is -
 (A) 108 (B) 207 (C) 669 (D) 729
- Q.22** If $\frac{5^x}{125} = 1$, then x is equal to -
 (A) 5 (B) 2 (C) 0 (D) 3
- Q.23** If $3^x - 3^{x-1} = 18$, then the value of x^x is -
 (A) 3 (B) 8 (C) 27 (D) 216
- Q.24** If $2^x - 2^{x-1} = 16$, then the value of x^2 is -
 (A) 4 (B) 9 (C) 16 (D) 25
- Q.25** If x and y are non-zero rational unequal numbers, then $\frac{(x+y)^2 - (x-y)^2}{x^2y - xy^2}$ is equal to -
 (A) $\frac{1}{xy}$ (B) $\frac{1}{x-y}$ (C) $\frac{4}{x-y}$ (D) $\frac{2}{x-y}$
- Q.26** If $\frac{x}{(b-c)(b+c-2a)} = \frac{y}{(c-a)(c+a-2b)}$
 $= \frac{z}{(a-b)(a+b-2c)}$, the value of $(x + y + z)$ is -
 (A) $a + b + c$ (B) $a^2 + b^2 + c^2$
 (C) 0 (D) indeterminate

- Q.27** Let $\frac{a}{b} - \frac{b}{a} = x : y$. If $(x - y) = \left(\frac{a}{b} + \frac{b}{a}\right)$, then x is equal to -
(A) $\frac{a+b}{a}$ (B) $\frac{a+b}{b}$
(C) $\frac{a-b}{a}$ (D) None of these
- Q.28** If $(x - 2)$ is a factor of $(x^2 + 3qx - 2q)$, then the value of q is -
(A) 2 (B) -2 (C) 1 (D) -1
- Q.29** If $x^3 + 6x^2 + 4x + k$ is exactly divisible by $(x + 2)$, then the value of k is -
(A) -6 (B) -7 (C) -8 (D) -10
- Q.30** Let $f(x) = x^3 - 6x^2 + 11x - 6$. Then, which one of the following is not factor of $f(x)$?
(A) $x - 1$ (B) $x - 2$ (C) $x + 3$ (D) $x - 3$
- Q.31** The polynomial $(x^4 - 5x^3 + 5x^2 - 10x + 24)$ has a factor as -
(A) $x + 4$ (B) $x - 2$
(C) $x + 2$ (D) None of these
- Q.32** Which of the following statements are correct ?
1. $x + 3$ is a factor of $x^3 + 2x^2 + 3x + 18$
2. $x + 2$ is a factor of $x^3 + 2x^2 - x - 2$
3. $x + 1$ is a factor of $x^3 + x^2 - 4x - 4$
4. $x - 2$ is a factor of $2x^3 - 3x + 4$
(A) 2, 3, 4 (B) 1, 3, 4
(C) 1, 2, 4 (D) 1, 2, 3
- Q.33** $(x^{29} - x^{25} + x^{13} - 1)$ is divisible by -
(A) both $(x - 1)$ & $(x + 1)$
(B) $(x - 1)$ but not by $(x + 1)$
(C) $(x + 1)$ but not by $(x - 1)$
(D) neither $(x - 1)$ nor $(x + 1)$
- Q.34** Value of k for which $(x - 1)$ is a factor of $(x^3 - k)$ is -
(A) -1 (B) 1 (C) 8 (D) -8
- Q.35** If $x^{100} + 2x^{99} + k$ is divisible by $(x + 1)$, then the value of k is -
(A) 1 (B) -3 (C) 2 (D) -2
- Q.36** If $(x^3 - 5x^2 + 4p)$ is divisible by $(x + 2)$, then the value of p is -
(A) 7 (B) -2 (C) 3 (D) -7
- Q.37** One of the factors of $3x^3 + x^2 - 12x - 4$ is -
(A) $3x - 2$ (B) $3x + 2$
(C) $3x - 1$ (D) $3x + 1$
- Q.38** Consider the following expressions :
1. $3x^2 - 48$ 2. $x^2 + 7x + 12$
3. $x^2 - x - 20$

The expression $(x + 4)$ would divide -

- (A) 1 & 2 (B) 2 & 3
(C) 1 & 3 (D) 1, 2 & 3

- Q.39** If $(x - a)$ is a factor of $(x^3 - 3x^2a + 2a^2x + b)$, then the value of b is -
(A) 0 (B) 2 (C) 1 (D) 3
- Q.40** If $x^3 + 3x^2 + 4x + k$ contains $(x + 6)$ as a factor, the value of k is -
(A) 66 (B) 33 (C) 132 (D) 36
- Q.41** If $(x + 2)$ and $(x - 1)$ are the factors of $(x^3 + 10x^2 + mx + n)$, the values of m and n are -
(A) $m = 5, n = -3$ (B) $m = 17, n = -8$
(C) $m = 7, n = -18$ (D) $m = 23, n = -19$
- Q.42** On dividing $(x^3 - 6x + 7)$ by $(x + 1)$, the remainder is -
(A) 2 (B) 12 (C) 0 (D) 7
- Q.43** If $(x^5 - 9x^2 + 12x - 14)$ is divided by $(x - 3)$, the remainder is -
(A) 184 (B) 56 (C) 2 (D) 1
- Q.44** When $(x^4 - 3x^3 + 2x^2 - 5x + 7)$ is divided by $(x - 2)$, the remainder is -
(A) 3 (B) -3 (C) 2 (D) 0
- Q.45** If $5x^3 + 5x^2 - 6x + 9$ is divided by $(x + 3)$, the remainder is -
(A) 135 (B) -135 (C) 63 (D) -63
- Q.46** If $(x^{11} + 1)$ is divided by $(x + 1)$, the remainder is -
(A) 0 (B) 2 (C) 11 (D) 12
- Q.47** The value of expression $(16x^2 + 24x + 9)$ for $x = -\frac{3}{4}$ is -
(A) 2 (B) 1 (C) 0 (D) -1
- Q.48** If $2x^3 + 5x^2 - 4x - 6$ is divided by $2x + 1$, the remainder is -
(A) $-\frac{13}{2}$ (B) 3 (C) -3 (D) 6
- Q.49** If $x^3 + 5x^2 + 10k$ leaves remainder $-2x$ when divided by $x^2 + 2$, then the value of k is -
(A) -2 (B) -1 (C) 1 (D) 2
- Q.50** When $(x^3 - 2x^2 + px - q)$ is divided by $(x^2 - 2x - 3)$, the remainder is $(x - 6)$. The values of p and q are -
(A) $p = -2, q = -6$ (B) $p = 2, q = -6$
(C) $p = -2, q = 6$ (D) $p = 2, q = 6$
- Q.51** The sum of $(x^2 + 1)$ and the reciprocal of $(x^2 - 1)$ is -
(A) $2x^2$ (B) $\frac{x^4}{x^2 - 1}$
(C) $\frac{x^2 - 2x + 2}{x^2 - 1}$ (D) $\frac{2x^2}{x^4 - 1}$
- Q.52** For making $(x^4 - 11x^2y^2 + y^4)$ a perfect square, the expression to be added is -

- (A) $5x^2y^2$ (B) $9x^2y^2$
 (C) $-5x^2y^2$ (D) $7x^2y^2$

Q.53 $(x^4 + 5x^3 + 6x^2)$ is equal to -

- (A) $x(x+3)(x^2+2)$ (B) $x^2(x+3)(x+2)$
 (C) $x^2(x-2)(x-3)$ (D) $x(x^2+3)(x+2)$

Q.54 If $(x^{3/2} - xy^{1/2} + x^{1/2}y - y^{3/2})$ is divided by $(x^{1/2} - y^{1/2})$, then the quotient is -

- (A) $x + y$ (B) $x - y$
 (C) $x^{1/2} + y^{1/2}$ (D) $x^2 - y^2$

Q.55 The factors of $(x^4 + 625)$ are -

- (A) $(x^2 - 25), (x^2 + 25)$
 (B) $(x^2 + 25), (x^2 + 25)$
 (C) $(x^2 - 10x + 25), (x^2 + 5x + 25)$
 (D) do not exist

Q.56 The factors of $\left(\frac{4}{x^4} + \frac{x^4}{81}\right)$ are -

- (A) not possible
 (B) $\left(\frac{2}{x^2} + \frac{x^2}{9}\right)\left(\frac{2}{x^2} - \frac{x^2}{9}\right)$
 (C) $\left(\frac{2}{x^2} + \frac{x^2}{9} + \frac{2}{3}\right)\left(\frac{2}{x^2} - \frac{x^2}{9} + \frac{2}{3}\right)$
 (D) $\left(\frac{2}{x^2} + \frac{x^2}{9} + \frac{2}{3}\right)\left(\frac{2}{x^2} + \frac{x^2}{9} - \frac{2}{3}\right)$

Q.57 The factors of $(x^2 - 1 - 2a - a^2)$ are -

- (A) $(x - a + 1)(x - a - 1)$
 (B) $(x + a - 1)(x - a + 1)$
 (C) $(x + a + 1)(x - a - 1)$
 (D) None of these

Q.58 The factors of $(x^2 - 8x - 20)$ are -

- (A) $(x + 10)(x - 2)$ (B) $(x - 10)(x + 2)$
 (C) $(x - 5)(x + 4)$ (D) $(x + 5)(x - 4)$

Q.59 The factors of $(x^2 - xy - 72y^2)$ are -

- (A) $(x - 8y)(x + 9y)$ (B) $(x - 9y)(x + 8y)$
 (C) $(x - y)(x + 72y)$ (D) $(x - 6y)(x + 12y)$

Q.60 The factors of $(x^2 - 11xy - 60y^2)$ are -

- (A) $(x + 15y)(x - 4y)$
 (B) $(x - 15y)(x + 4y)$
 (C) $(15x + y)(4x - y)$

(D) $(x - 2)^2 (x + 1)$

Q.70 The factors of $(x^4 + 4)$ are -

(A) $(x^2 + 2)^2$

(B) $(x^2 + 2)(x^2 - 2)$

(C) $(x^2 + 2x + 2)(x^2 - 2x + 2)$

(D) None of these

Q.71 $(x + y)^3 - (x - y)^3$ can be factorized as -

(A) $2y(3x^2 + y^2)$ (B) $2x(3x^2 + y^2)$

(C) $2y(3y^2 + x^2)$ (D) $2x(x^2 + 3y^2)$

Q.72 The factors of $(x^3 + 8y^3)$ are -

(A) $(x + 2y)(x^2 - 2xy + 4y^2)$

(B) $(x + 2y)(x^2 + 2xy + 4y^2)$

(C) $(x + 2y)(x - 2y)^2$

(D) $(x + 2y)^3$

Q.73 If one of the two factors of an expression which is the difference of two cubes is $(x^4 + x^2y + y^2)$, then the other factor would be -

(A) $x + y$ (B) $x - y$

(C) $x^2 + y$ (D) $x^2 - y$

Q.74 The factors of $\left(x^2 + \frac{1}{x^2} - 3\right)$ are -

(A) $\left(x + \frac{1}{x} - 1\right)\left(x - \frac{1}{x} - 1\right)$

(B) $\left(x - \frac{1}{x} - 1\right)\left(x + \frac{1}{x} + 1\right)$

(C) $\left(x - \frac{1}{x} + 1\right)\left(x - \frac{1}{x} - 1\right)$

(D) $\left(x + \frac{1}{x} - 1\right)\left(x - \frac{1}{x} + 1\right)$

Q.75 The factors of $x^2(y - z) + y^2(z - x) + z^2(x - y)$ are -

(A) $(x + y + z)(xy + yz + zx)$

(B) $(x - y)(y - z)(z - x)$

(C) $(x - y - z)(xy - yz - zx)$

(D) $(x - y)(y - z)(x - z)$

Q.76 The factors of $x^3 - 7x + 6$ are -

(A) $x(x - 6)(x - 1)$

(B) $(x^2 - 6)(x - 1)$

(C) $(x + 1)(x + 2)(x - 3)$

(D) $(x - 1)(x + 3)(x - 2)$

Q.77 The factors of $(x - y)^3 + (y - z)^3 + (z - x)^3$ are -

(A) $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

(B) $(x - y - z)(x^2 + y^2 + z^2 + xy + yz + zx)$

(C) $(x - y)(y - z)(z - x)$

(D) $3(x - y)(y - z)(z - x)$

Q.78 The factors of $(8a^3 + b^3 - 6ab + 1)$ are -

- (A) $(2a + b - 1)(4a^2 + b^2 + 1 - 3ab - 2a)$
 (B) $(2a - b + 1)(4a^2 + b^2 - 4ab + 1 - 2a + b)$
 (C) $(2a + b + 1)(4a^2 + b^2 + 1 - 2ab - b - 2a)$
 (D) $(2a - 1 + b)(4a^2 + 1 - 4a - b - 2ab)$

Q.79 The factors of $(x^3 - 1 + y^3 + 3xy)$ are -

- (A) $(x - 1 + y)(x^2 + 1 + y^2 + x + y - xy)$
 (B) $(x + y + 1)(x^2 + y^2 + 1 - xy - y - x)$
 (C) $(x - 1 + y)(x^2 - 1 - y^2 + x + y + xy)$
 (D) $3(x + y - 1)(x^2 + y^2 - 1)$

Q.80 The factors of $[(x + y)^3 - x - y]$ are -

- (A) $(x - y)(x - y + 1)(x - y - 1)$
 (B) $(x - y)(x + y + 1)(x + y - 1)$
 (C) $(x + y)(x - y + 1)(x + y - 1)$
 (D) $(x + y)(x + y - 1)(x + y + 1)$

Q.81 The factors of $(x^4 - 13x^2 + 36)$ are -

- (A) $(x - 1)(x + 1)(x - 6)(x + 6)$
 (B) $(x - 3)(x + 3)(x - 2)(x + 2)$
 (C) $(x^2 + 9)(x - 2)(x + 2)$
 (D) $(x^2 + 4)(x + 3)(x - 3)$

Q.82 The value of :

$$(a^{1/8} + a^{-1/8})(a^{1/8} - a^{-1/8})(a^{1/4} + a^{-1/4})(a^{1/2} + a^{-1/2}) \text{ is -}$$

- (A) $(a + a^{-1})$ (B) $(a - a^{-1})$
 (C) $(a^2 - a^{-2})$ (D) $(a^{1/2} - a^{-1})$

Q.83 What must be added to $\frac{1}{x}$ to make it equal to x ?

- (A) $\frac{x-1}{x+1}$ (B) $\frac{x+1}{x-1}$
 (C) $\frac{x^2+1}{x}$ (D) $\frac{x^2-1}{x}$

Q.84 If $a + b + c = 0$, then $a^2 + b^2 + c^2$ is -

- (A) $-4(ab + bc + ca)$ (B) $-2(ab + bc + ca)$
 (C) 0 (D) $2a^2 - 2bc$

Q.85 If $(5x^2 + 14x + 2)^2 - (4x^2 - 5x + 7)^2$ is divided by $(x^2 + x + 1)$, then quotient q and remainder r are given by -

- (A) $q = (x^2 + 19x - 5), r = 1$
 (B) $q = 9(x^2 + 19x - 5), r = 0$
 (C) $q = (x^2 + 19x - 5), r = 0$
 (D) $q = 9(x^2 + 19x - 5), r = 1$

Q.86 The polynomial (in x) : $x^6 + 18x^3 + 125$, admits-

- (A) No factor of degree between 2 and 5 (B) $x^2 - 3x + 5$ as a factor

(C) $x + 1$ as a factor(D) $x - 125$ as a factor

Q.87 When $x^3 - 6x^2 + 12x - 4$ is divided by $x - 2$, the remainder is -

(A) 4 (B) 0 (C) 5 (D) 6

Q.88 The value of x satisfying the equation

$$x^2 + p^2 = (q - x)^2 \text{ is } =$$

(A) $\frac{p^2 - q^2}{2}$ (B) $\frac{q^2 - p^2}{2q}$ (C) $\frac{q^2 - p^2}{2}$ (D) $\frac{p^2 - q^2}{2q}$

Q.89 If $2x^2 + xy - 3y^2 + x + ay - 10 = (2x + 3y + b)(x - y - 2)$, the value of a and b are -

(A) 11 and 5 (B) 1 and -5

(C) -1 and -5 (D) -11 and 5

Q.90 If $a - b = 3$ and $a^3 - b^3 = 117$, then $a + b$ is equal to -

(A) 5 (B) 7 (C) 9 (D) 11

Q.91 The value of n for which the expression $9x^4 - 12x^2 + nx^2 - 8x + 4$ becomes a perfect square, is -

(A) 12 (B) 16

(C) 18 (D) 24

Q.92 Which one of the following is not a factor of $x^3 + 2x^2 + x$?

(A) $x + 2$ (B) $x + 1$ (C) x (D) $x(x + 1)$

Q.93 When $x^{13} + 1$ is divided by $x + 1$, the remainder is -

(A) -1 (B) 0 (C) 1 (D) 2

Q.94 When $x^3 + 2x^2 + 2x - 4$ and $x^3 + 2x^2 - 3x + 6$ are divided by $x - 2$, the remainders are R_1 and R_2 respectively. Which of the following statements is true for R_1 and R_2 ?

(A) $R_1 = 2R_2$ (B) $2R_1 = R_2$ (C) $R_1 = R_2$ (D) $R_1 + R_2 = 0$

One or More than One Correct Choice Type Questions

Q.95 If $a^2(b - c) + b^2(c - a) + c^2(a - b) = k(a - b)(b - c)(c - a)$, then which of these are not true -

(A) $k = 1$ (B) $k = -1$ (C) $k = 2$ (D) $k = -2$

Q.96 If $xy(x - y) + yz(y - z) + zx(z - x) = k(x - y)(y - z)(z - x)$, then which of these are not true :

(A) $k = 1$ (B) $k = 0$ (C) $k = -1$ (D) $k = -2$

Q.97 For any number $a < 1$, $a + \frac{1}{a}$ is always -

(A) less than 2 (B) greater than 2

- (C) less than 3 (D) greater than 3

Passage Based Questions

PASSAGE 1 (Q.98 TO Q. 102)

Since polynomials are nothing but algebraic expressions of special form, they are added, subtracted, multiplied and divided just like algebraic expressions. However, because of their special form, it is possible to organize the working in a convenient manner. If among terms only numerical coefficients differ they are called like terms. Like terms can be added. Only like terms can be subtracted. The products can be speedily calculated by finding the sums of like powers of x or like terms.

- Q.98** If $a - \frac{1}{2}b + c$, $b - \frac{1}{2}c + a$, $c - \frac{1}{2}a + b$ are added together and the result is subtracted from 0, then-
- (A) $\frac{3}{2}a + \frac{3}{2}b + \frac{3}{2}c$
 (B) $\frac{3}{2}a - \frac{3}{2}b + \frac{3}{2}c$
 (C) $-\frac{3}{2}a - \frac{3}{2}b - \frac{3}{2}c$
 (D) None of these
- Q.99** Subtract $5a - 3b - 2c$ from $6a - 5b + 7c$
- (A) $a - 2b + 9c$ (B) $a + 2b + 9c$
 (C) $a - 2b - 9c$ (D) $-a - 2b + 9c$
- Q.100** Subtract $1 - 3x + 3x^2 - x^3$ from $x^3 - 3x^2 + 3x - 1$, then
- (A) $2x^3 + 6x^2 + 6x - 2$
 (B) $2x^3 - 6x^2 + 6x - 2$
 (C) $2x^3 - 6x^2 - 6x + 2$
 (D) $2x^3 + 6x^2 + 6x + 2$
- Q.101** Multiply $3x^2 + 5x - 1$ by $4x - 3$, then -
- (A) $12x^3 + 11x^2 - 19x + 3$
 (B) $12x^3 - 11x^2 + 19x + 3$
 (C) $12x^3 - 11x^2 - 19x + 3$
 (D) $12x^3 + 11x^2 + 19x + 3$
- Q.102** The coefficient of x^2 in the product of $(2x^2 - 3ax + a^2)(3ax - 2a^2 - x^2)$ is equal to :
- (A) $-14a^2$ (B) $+14a^2$
 (C) $13a^2$ (D) None of these

PASSAGE 2 (Q.103 TO Q. 107)

Some times a given expression is not a quadratic polynomial but the product of the variables in the first and third term is equal to the square of the variables in the middle term. In such cases the given expression can be factorized by splitting the middle term such that their sum is equal to the middle term and the product is equal to the product of first and last term.

- Q.103** Factorization of $x^2y^2 - xy - 72$ is :
- (A) $(xy - 9)(xy + 8)$ (B) $(xy + 9)(xy + 8)$
 (C) $(xy - 9)(xy - 8)$ (D) none of these

Q.104 Factorization of $x^4 - 14x^2y^2 - 51y^4$ is -

(A) $(x + \sqrt{17}y)(x + \sqrt{17}y)(x^2 + 3y^2)$

(B) $(x - \sqrt{17}y)(x + \sqrt{17}y)(x^2 + 3y^2)$

(C) $(x - \sqrt{17}y)(x + \sqrt{17}y)(x^2 - 2y^2)$

(D) $(x - \sqrt{17}y)(x + \sqrt{17}y)(x^2 + 2y^2)$

Q.105 If $x^2 + px + q = (x + a)(x + b)$, then factorization of $x^2 + pxy + qy^2$ is

(A) $(x + ay)(x - by)$ (B) $(x - ay)(x + by)$

(C) $(x + ay)(x + by)$ (D) None of these

Q.106 Factorization of $x^2 + 2x - 63$ is -

(A) $(x + 9)(x - 7)$ (B) $(x - 9)(x + 7)$

(C) $(x - 9)(x - 7)$ (D) $(x + 9)(x + 7)$

Q.107 Factorization of $x^2 - 5x + 6$ is

(A) $(x - 2)(x - 3)$ (B) $(x + 2)(x + 3)$

(C) $(x + 2)(x - 3)$ (D) $(x - 2)(x + 3)$

ANSWER KEY

Q.No	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Ans.	C	C	B	A	D	B	A	A	D	C	B	C	D	C	A	C	B	B	C	D	A	D
Q.No	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
Ans.	C	D	C	C	D	D	C	C	B	D	B	B	A	A	D	D	A	C	C	B	A	B
Q.No	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66
Ans.	D	A	C	C	C	C	B	B	B	A	D	D	C	B	B	B	A	A	A	B	D	D
Q.No	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88
Ans.	B	B	C	C	A	A	D	C	D	D	D	C	A	D	B	B	D	B	B	B	A	B
Q.No	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107			
Ans.	D	B	B	A	B	C	A,C,D	A,B,D	B	C	A	B	A	A	A	B	C	A	A			

HINTS & SOLUTION

A. SINGLE CHOICE QUESTIONS :

18.[B] Using $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$,

we get : $x^{1/3} + y^{1/3} + z^{1/3} = 0$

$\Rightarrow x + y + z = 3x^{1/3} y^{1/3} z^{1/3}$

$\Rightarrow (x + y + z)^3 = 27xyz$.

19.[C] $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

20.[D] $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$;

$\therefore \frac{a^3}{abc} + \frac{b^3}{abc} + \frac{c^3}{abc} = 3$

or $\frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} = 3$.

21.[A] $(x^3 + y^3 + z^3 - 3xyz)$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$= (x + y + z)[(x + y + z)^2 - 3(xy + yz + zx)]$

$= 9 \times (81 - 3 \times 23) = (2 \times 12) = 108$.

23.[C] $3^x - 3^{x-1} = 18 \Rightarrow 3^x - \frac{3^x}{3} = 18$

$\Rightarrow a - \frac{a}{3} = 18$, where $a = 3^x \Rightarrow 2a = 54$

$\Rightarrow a = 27 \Rightarrow 3^x = 3^3 \Rightarrow x = 3$.

$\therefore x^x = 3^3 = 27$.

24.[D] $2^x - 2^{x-1} = 16 \Rightarrow 2^x - \frac{2^x}{2} = 16 \Rightarrow a - \frac{a}{2} = 16$

$\Rightarrow \frac{a}{2} = 16 \Rightarrow a = 32$

$\Rightarrow 2^x = 2^5 \Rightarrow x = 5 \therefore x^2 = 5^2 = 25$.

25.[C] Given Exp. $= \frac{4xy}{xy(x-y)} = \frac{4}{(x-y)}$
 $[\because (a+b)^2 - (a-b)^2 = 4ab]$

$$27.[D] \frac{x}{y} = \frac{a^2 - b^2}{ab} \Rightarrow \frac{x+y}{x-y} = \frac{a^2 - b^2 + ab}{a^2 - b^2 - ab}$$

$$\text{Also, } (x - y) = \frac{a^2 + b^2}{ab} \quad \dots(i)$$

$$\therefore x + y = \frac{(a^2 - b^2 + ab)}{(a^2 - b^2 - ab)} \cdot \frac{(a^2 + b^2)}{ab} \quad \dots(ii)$$

Adding (i) and (ii) we get :

$$\begin{aligned} x &= \frac{1}{2} \left(\frac{a^2 + b^2}{ab} \right) \cdot \left[1 + \frac{a^2 - b^2 + ab}{a^2 - b^2 - ab} \right] \\ &= \frac{(a^4 - b^4)}{(a^2 - b^2 - ab)} \end{aligned}$$

29.[C] Since $(x + 2)$ is a factor, so $x = -2$ will make it zero.

$$\therefore (-2)^3 + 6(-2)^2 + 4(-2) + k = 0$$

$$\text{or } k = -8$$

35.[A] Since $(x + 2)$ is a factor, $x = -2$ will make the given expression zero.

$$\therefore -8 - 20 + 4p = 0 \Rightarrow p = 7.$$

41.[C] Clearly, $x = -2$ as well as $x = 1$ will make the given expression zero. So,

$$2m - n = 32 \text{ \& } m + n = -11.$$

Solving these equations, we get : $m = 7, n = -18$

42.[B] Let $f(x) = x^3 - 6x + 7$. This when divided by $(x + 1)$ gives remainder,

$$f(-1) = (-1)^3 - 6 \times (-1) + 7 = 12.$$

43.[A] Let $f(x) = x^5 - 9x^2 + 12x - 14$. This when divided by $(x - 3)$ gives remainder

$$f(3) = 3^5 - 9 \times 3^2 + 12 \times 3 - 14 = 184.$$

44.[B] Remainder is

$$f(2) = 2^4 - 3 \times 2^3 + 2 \times 2^2 - 5 \times 2 + 7 = -3.$$

49.[C] On actually dividing $x^3 + 5x^2 + 10k$ by $(x^2 + 2)$, the remainder obtained is $-2x + 10k - 10$.

$$\therefore -2x + 10k - 10 = -2x \text{ or } k = 1.$$

55.[D] Factors of $(x^4 + 625)$ do not exist.

$$56.[D] \frac{4}{x^4} + \frac{x^4}{81} = \left(\frac{2}{x^2} \right)^2 + \left(\frac{x^2}{9} \right)^2 + 2 \times \frac{2}{x^2} \times \frac{x^2}{9} - \frac{4}{9}$$

$$= \left(\frac{2}{x^2} + \frac{x^2}{9} \right)^2 - \left(\frac{2}{3} \right)^2$$

$$= \left(\frac{2}{x^2} + \frac{x^2}{9} + \frac{2}{3} \right) \left(\frac{2}{x^2} + \frac{x^2}{9} - \frac{2}{3} \right)$$

73.[D] $x^4 + x^2y + y^2 = [(x^2)^2 + x^2y + (y)^2]$

$$= (a^2 + ab + b^2), \text{ where } a = x^2, b = y$$

$$\therefore \text{The other factor is } (a - b) = (x^2 - y).$$

82.[B] Using $(a + b)(a - b) = a^2 - b^2$, we get :

$$(a^{1/8} + a^{-1/8})(a^{1/8} - a^{-1/8})(a^{1/4} + a^{-1/4})(a^{1/2} + a^{-1/2})$$

$$= (a^{1/4} - a^{-1/4})(a^{1/4} + a^{-1/4})(a^{1/2} + a^{-1/2})$$

$$= (a^{1/2} - a^{-1/2})(a^{1/2} + a^{-1/2}) = (a^1 - a^{-1}).$$

84.[B] $(a + b + c)^2 = 0$

$$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$$

$$\Rightarrow a^2 + b^2 + c^2 = -2(ab + bc + ca)$$

85.[B] Using $A^2 - B^2 = (A + B)(A - B)$, we get :

Given Exp.

$$= (5x^2 + 14x + 2 + 4x^2 - 5x + 7)(5x^2 + 14x + 2 - 4x^2 + 5x - 7)$$

$$= (9x^2 + 9x + 9)(x^2 + 19x - 5)$$

$$= 9(x^2 + 19x - 5)(x^2 + x + 1)$$

So, $\frac{\text{Given Exp.}}{(x^2 + x + 1)} = 9(x^2 + 19x - 5);$

$$\therefore q = 9(x^2 + 19x - 5) \text{ \& } r = 0$$

86.[B] putting

$$x^2 = 3x - 5 \text{ in } x^6 + 18x^3 + 125$$

We get $(3x-5)^3 + 18x(3x-5) + 125$

$$= 27x^3 - 125 - 45x(3x-5) + 54x^2 - 90x + 125$$

$$= 27x^3 - 135x^2 + 225x + 54x^2 - 90x$$

$$= 27x^3 - 81x^2 + 135x$$

$$= 27x(x^2 - 3x + 5) = 27x(3x - 5 - 3x + 5) = 0$$

$$\therefore x^2 - 3x + 5 \text{ is factor}$$

87.[A]

$$\begin{array}{r} x^2 + 4x + 4 \\ x - 2 \quad x^3 - 6x^2 + 12x - 4 \\ \quad x^3 - 2x^2 \\ \quad - + \\ \quad - 4x^2 + 12x \\ \quad \quad 4x^2 - 8x \\ \quad \quad + \quad + \quad + \\ \quad \quad \quad 4x - 4 \\ \quad \quad \quad + 4x - 8 \\ \quad \quad \quad - \quad + \\ \quad \quad \quad \quad 4 \end{array}$$

Remainder = 4

88.[B] $x^2 + p^2 = q^2 - 2qx + x^2$

or, $x^2 + 2qx - x^2 = q^2 - p^2$

or, $2qx = q^2 - p^2$

or, $x = \frac{q^2 - p^2}{2q}$

89.[D] $\therefore 2x^2 + xy - 3y^2 + x + ay - 10$

$$= (2x + 3y + b)(x - y - 2)$$

$$= 2x^2 + xy - 3y^2 + (b - 4)x - (b + 6)y - 2b$$

Compare the coefficients of x , y and constant on both sides.

$$b - 4 = 1 \quad \Rightarrow \quad b = 5$$

$$-(b + 6) = a \quad \Rightarrow \quad a = -11$$

$$2b = 10 \quad \Rightarrow \quad b = 5$$

$$\therefore a = -11 \text{ and } b = 5$$

90.[B] $a - b = 3$

$$\Rightarrow a = b + 3 \quad \dots(i)$$

$$\text{and } a^3 - b^3 = 117$$

$$\Rightarrow (a - b)(a^2 + ab + b^2) = 117 \quad \dots(ii)$$

Divided (ii) by (i),

$$\therefore a^2 + ab + b^2 = \frac{117}{3} = 39 \quad \dots(iii)$$

Put the value of b in eq. (iii),

$$\therefore a^2 + a(a - 3) + (a - 3)^2 = 39$$

$$\Rightarrow 3a^2 - 9a + 9 = 39$$

$$\Rightarrow a^2 - 3a - 10 = 0$$

$$\Rightarrow (a + 2)(a - 5) = 0$$

$$\therefore a = -2 \text{ or } 5$$

$$\text{and } b = -5 \text{ or } 2$$

$$\therefore a + b = 5 + 2 = 7$$

92.[A] $x + 2$ is not a factor of $p(x) = x^3 + 2x^2 + x$

For $x + 2$ to be a factor of $p(x)$; $p(-2)$ should be zero by the factor theorem.

$$\text{Now } p(-2) = (-2)^3 + 2(-2)^2 + (-2)$$

$$= -8 + 8 - 2$$

$$= -2 \neq 0$$

94.[C]

$$\begin{array}{r} x^2 + 4x + 10 \\ x - 2 \quad x^3 + 2x^2 + 2x - 4 \\ \quad x^3 - 2x^2 \\ \quad - \quad + \\ \quad \quad 4x^2 + 2x - 4 \\ \quad \quad 4x^2 - 8x \\ \quad \quad - \quad + \\ \quad \quad \quad 10x - 4 \\ \quad \quad \quad 10x - 20 \\ \quad \quad \quad - \quad + \\ \quad \quad \quad \quad 16 = R_1 \end{array}$$

$$\begin{array}{r} x^2 + 4x + 5 \\ x - 2 \quad x^3 + 2x^2 - 3x + 6 \\ \quad x^3 - 2x^2 \\ \quad - \quad + \\ \quad \quad 4x^2 - 3x + 6 \end{array}$$

$$\begin{array}{r}
 4x^2 - 8x \\
 - \quad + \\
 5x + 6 \\
 5x - 10 \\
 - \quad + \\
 16 = R_2
 \end{array}$$

So $R_1 = R_2$, (C) is the correct answer.

B. ONE OR MORE THAN ONE CORRECT CHOICE TYPE QUESTIONS :

95.[A,C,D] $a^2(b - c) + b^2(c - a) + c^2(a - b)$
 $= k(a - b)(b - c)(c - a)$
 R.H.S. $= k(a - b)(b - c)(c - a)$
 $= kabc - ka^2b - kac^2 + ka^2c - kcb^2 + kb^2a$
 $\quad + kbc^2 - kabc$
 $= a^2(-kb + kc) + b^2(-kc + ka) + c^2(-ka + kb)$
 \therefore compare the coefficient of a^2, b^2, c^2 in L.H.S. and R.H.S.
 $b - c = -kb + kc \Rightarrow b - c = -k(b - c)$
 or $k = -\frac{(b-c)}{(b-c)} = -1$
 Similarly $c - a = -kc + ka \Rightarrow k = -1$
 $a - b = -ka + kb \Rightarrow k = -1$
 \therefore the correct answer are (A), (C) and (D)

96.[A,B,D] $xy(x - y) + yz(y - z) + zx(z - x)$
 $= k(x - y)(y - z)(z - x)$
 $x^2y - xy^2 + y^2z - yz^2 + z^2x - zx^2$
 $= x^2(-ky + kz) + y^2(kx - kz) + z^2(-kx + ky)$
 $\Rightarrow x^2(y - z) + y^2(z - x) + z^2(x - y)$
 $= x^2(-ky + kz) + y^2(kx - kz) + z^2(-kx + ky)$
 \Rightarrow compare the coefficients of x^2, y^2 and z^2
 $y - z = -k(y - z) \Rightarrow k = -1$
 similarly for y^2 and $z^2 \Rightarrow k = -1$
 \therefore (A), (B) and (D) are the the correct answers.

97.[B] If we take $a = 0.5$

$$\text{then } a + \frac{1}{a} = 0.5 + \frac{1}{0.5} = 2.5$$

Which is greater than 2

C. PASSAGE BASED QUESTIONS :

98.[C] $a - \frac{1}{2}b + c$
 $a + b - \frac{1}{2}c$
 $-\frac{1}{2}a + b + c$

 $\frac{3}{2}a + \frac{3}{2}b + \frac{3}{2}c$

$$\begin{aligned} \therefore 0 - \left(\frac{3}{2} a + \frac{3}{2} b + \frac{3}{2} c\right) \\ = -\frac{3}{2} a - \frac{3}{2} b - \frac{3}{2} c \end{aligned}$$

$$\begin{array}{r} 99.[A] \quad 6a - 5b + 7c \\ \quad 5a - 3b - 2c \\ \quad - \quad + \quad + \\ \hline \end{array}$$

$$a - 2b + 9c$$

$$\begin{array}{r} 100.[B] \quad x^3 - 3x^2 + 3x - 1 \\ \quad -x^3 + 3x^2 - 3x + 1 \\ \quad + \quad - \quad + \quad - \\ \hline \end{array}$$

$$2x^3 - 6x^2 + 6x - 2$$

$$\begin{array}{r} 101.[A] \quad 3x^2 + 5x - 1 \\ \quad 4x - 3 \\ \hline \end{array}$$

$$\begin{array}{r} 12x^3 + 20x^2 - 4x \\ \quad - 9x^2 \quad - 15x + 3 \\ \hline \end{array}$$

$$12x^3 + 11x^2 - 19x + 3$$

102.[A] Rewriting the two expressions in descending powers of x -
 $(2x^2 - 3ax + a^2)(-x^2 + 3ax - 2a^2)$

The required coefficient of

$$\begin{aligned} x^2 &= 2(-2a^2) + (-3a)(+3a) + a^2(-1) \\ &= -4a^2 - 9a^2 - a^2 = -14a^2 \end{aligned}$$

103.[A] In order to factorize

$x^2y^2 - xy - 72$, we have to find two numbers p and q such that $p + q = -1$ and $pq = -72$

Clearly $-9 + 8 = -1$ and $-9 \times 8 = -72$

So we write the middle term $-xy$ of $x^2y^2 - xy - 72$ as $-9xy + 8xy$, so that

$$x^2y^2 - xy - 72 = x^2y^2 - 9xy + 8xy - 72$$

$$= (x^2y^2 - 9xy) + (8xy - 72)$$

$$= xy(xy - 9) + 8(xy - 9)$$

$$= (xy - 9)(xy + 8)$$

104.[B] Here $p + q = -14$ and $pq = -51$

Clearly, $-17 + 3 = -14$ and $-17 \times 3 = -51$

$$\therefore -14x^2y^2 = -17x^2y^2 + 3x^2y^2$$

$$\text{So that } x^4 - 14x^2y^2 - 51y^4$$

$$= x^4 - 17x^2y^2 + 3x^2y^2 - 51y^4$$

$$\begin{aligned}
 &= x^2(x^2 - 17y^2) + 3y^2(x^2 - 17y^2) \\
 &= (x^2 - 17y^2)(x^2 + 3y^2) \\
 &= \{x^2 - (\sqrt{17}y)^2\}(x^2 + 3y^2) \\
 &= (x - \sqrt{17}y)(x + \sqrt{17}y)(x^2 + 3y^2)
 \end{aligned}$$

105.[C] We have,

$$\begin{aligned}
 x^2 + px + q &= (x + a)(x + b) \\
 \Rightarrow x^2 + px + q &= x^2 + x(a + b) + ab \\
 \text{on equating the coefficients of like powers of } x, &\text{ we get} \\
 p &= a + b \text{ and } q = ab \\
 \therefore x^2 + px + q &= x^2 + (a + b)xy + aby^2 \\
 &= (x^2 + axy) + (bxy + aby^2) \\
 &= x(x + ay) + by(x + ay) \\
 &= (x + ay)(x + by)
 \end{aligned}$$

106.[A] $p + q = 2$ and $pq = -63$

$$\begin{aligned}
 x^2 + 2x - 63 &= x^2 + (9x - 7x) - 63 \\
 &= (x^2 + 9x) - (7x + 63) \\
 &= x(x + 9) - 7(x + 9) \\
 &= (x + 9)(x - 7)
 \end{aligned}$$

107.[A] $x^2 - 5x + 6 = x^2 - 2x - 3x + 6$

$$\begin{aligned}
 &= x(x - 2) - 3(x - 2) \\
 &= (x - 2)(x - 3)
 \end{aligned}$$

