Real Numbers

- **Euclid's Division Lemma**

For any given positive integers a and b, there exists unique integers q and r such that
$a = bq + r$ where $0 \le r < b$

Note: If b divides a, then $r = 0$

Example 1:

For $a = 15$, $b = 3$, it can be observed that
$15 = 3 \times 5 + 0$
Here, $q = 5$ and $r = 0$
If b divides a, then $0 < r < b$

Example 2:

For $a = 20$, $b = 6$, it can be observed that $20 = 6 \times 3 + 2$
Here, $q = 6$, $r = 2$, $0 < 2 < 6$

- **Euclid's division algorithm**

Euclid's division algorithm is a series of well-defined steps based on "Euclid's division lemma", to give a procedure for calculating problems.

Steps for finding HCF of two positive integers *a* and *b* (*a* > *b*) by using Euclid's division algorithm:

Step 1: Applying Euclid's division lemma to a and b to find whole numbers q and r, such that $a = bq + r$, $0 \le r < b$
Step 2: If $r = 0$, then HCF $(a, b) = b$
If $r \neq 0$, then again apply division lemma to b and r
Step 3: Continue the same procedure till the remainder is 0. The divisor at this stage will be the HCF of a and b.

Note: HCF (a, b) = HCF (b, r)

Example:

Find the HCF of 48 and 88.

Solution:

Take $a = 88$, $b = 48$
Applying Euclid's division lemma, we get
$88 = 48 \times 1 + 40$ (Here, $0 \le 40 < 48$)

$48 = 40 \times 1 + 8$ (Here, $0 \leq 8 < 40$)
$40 = 8 \times 5 + 0$ (Here, $r = 0$)
HCF (48, 88) = 8

- For any positive integer a, b, **HCF (a, b) × LCM (a, b) = $a \times b$**

Example 1:
Find the LCM of 315 and 360 by the prime factorisation method. Hence, find their HCF.

Solution:
$315 = 3 \times 3 \times 5 \times 7 = 3^2 \times 5 \times 7$
$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$
$\text{LCM} = 3^2 \times 5 \times 7 \times 2^3 = 2520$
$$\therefore \text{HCF}\left(315, 360\right) = \frac{315 \times 360}{\text{LCM}(315, 360)} = \frac{315 \times 360}{2520} = 45$$

Example 2:
Find the HCF of 300, 360 and 240 by the prime factorisation method.

Solution:
$300 = 2^2 \times 3 \times 5^2$
$360 = 2^3 \times 3^2 \times 5$
$240 = 2^4 \times 3 \times 5$
$\text{HCF}(300, 360, 240) = 2^2 \times 3 \times 5 = 60$

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- **Using Euclid's division lemma to prove mathematical relationships**

Result 1:

Every positive even integer is of the form $2q$, while every positive odd integer is of the form $2q + 1$, where q is some integer.

Proof:

Let a be any given positive integer.

Take $b = 2$

By applying Euclid's division lemma, we have

$a = 2q + r$ where $0 \leq r < 2$

As $0 \leq r < 2$, either $r = 0$ or $r = 1$

If $r = 0$, then $a = 2q$, which tells us that a is an even integer.

If $r = 1$, then $a = 2q + 1$

It is known that every positive integer is either even or odd.

Therefore, a positive odd integer is of the form $2q + 1$.

Result 2:

Any positive integer is of the form $3q$, $3q + 1$ or $3q + 2$, where q is an integer.

Proof:

Let a be any positive integer.

Take $b = 3$

Applying Euclid's division lemma, we have

$a = 3q + r$, where $0 \leq r < 3$ and q is an integer
Now, $0 \leq r < 3$ Þ $r = 0, 1$, or 2
$\therefore a = 3q + r$
$\Rightarrow a = 3q + 0, a = 3q + 1, a = 3q + 2$
Thus, $a = 3q$ or $a = 3q + 1$ or $a = 3q + 2$, where q is an integer.

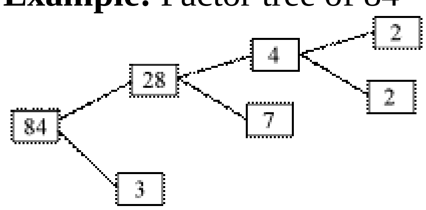
- Fundamental theorem of arithmetic states that very composite number can be uniquely expressed (factorised) as a product of primes apart from the order in which the prime factors occur.

Example: 1260 can be uniquely factorised as

2	1260
2	630
3	315
3	105
5	35
	7

$1260 = 2 \times 2 \times 3 \times 3 \times 5 \times 7$

Example: Factor tree of 84



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HCF (300, 360, 240) = $2^2 \times 3 \times 5 = 60$

- According to fundamental theorem of arithmetic, a number can be represented as the product of primes having a unique factorisation.

Example:
Check whether 15^n in divisible by 10 or not for any natural number n. Justify your answer.

Solution:
A number is divisible by 10 if it is divisible by both 2 and 5.
$15^n = (3 \,.\, 5)^n$
3 and 5 are the only primes that occur in the factorisation of 15^n
By uniqueness of fundamental theorem of arithmetic, there is no other prime except 3 and 5 in the factorisation of 15^n.
2 does not occur in the factorisation of 15^n.
Hence, 15^n is not divisible by 10.

- Every number of the form $\sqrt{p}$, where p is a prime number is called an irrational number. For example, $\sqrt{3}$, $\sqrt{11}$, $\sqrt{12}$ etc.

Theorem: If a prime number p divides a^2, then p divides a, where a is a positive integer.

Example:
Prove that $\sqrt{7}$ is an irrational number.

Solution:
If possible, suppose $\sqrt{7}$ is a rational number.
Then, $\sqrt{7} = \frac{p}{q}$, where p, q are integers, $q \neq 0$.
If HCF $(p, q) \neq 1$, then by dividing p and q by HCF (p, q), $\sqrt{7}$ can be reduced as
$\sqrt{7} = \frac{a}{b}$ where HCF $(a, b) = 1$ … (1)
$\Rightarrow \sqrt{7}b = a$
$\Rightarrow 7b^2 = a^2$
$\Rightarrow a^2$ is divisible by 7
$\Rightarrow a$ is divisible by 7 … (2)
$\Rightarrow a = 7c$, where c is an integer
$\therefore \sqrt{7}c = b$
$\Rightarrow 7b^2 = 49c^2$
$\Rightarrow b^2 = 7c^2$

$\Rightarrow b^2$ is divisible by 7
$\Rightarrow b$ is divisible by 7 … (3)
From (2) and (3), 7 is a common factor of a and b. which contradicts (1)

$\therefore \sqrt{7}$ is an irrational number.

Example:
Show that $\sqrt{12} - 6$ is an irrational number.

Solution:
If possible, suppose $\sqrt{12} - 6$ is a rational number.
Then $\sqrt{12} - 6 = \frac{p}{q}$ for some integers p, q (q ¹ 0)
Now,
$\sqrt{12} - 6 = \frac{p}{q}$
$\Rightarrow 2\sqrt{3} = \frac{p}{q} + 6$
$\Rightarrow \sqrt{3} = \frac{1}{2}\left(\frac{p}{q} + 6\right)$
As p, q, 6 and 2 are integers, $\frac{1}{2}\left(\frac{p}{q} + 6\right)$ is rational number, so is $\sqrt{3}$.
This conclusion contradicts the fact that $\sqrt{3}$ is irrational.
Thus, $\sqrt{12} - 6$ is an irrational number.

- Decimal expansion of a rational number can be of two types:

(i) Terminating
(ii) Non-terminating and repetitive
In order to find decimal expansion of rational numbers we use long division method.
For example, to find the decimal expansion of $\frac{1237}{25}$.
We perform the long division of 1237 by 25.

$$
\begin{array}{r|r}
 & 49.48 \\ \hline
25 & 1237.00 \\
 & \underline{100} \\
 & 237 \\
 & \underline{225} \\
 & 120 \\
 & \underline{100} \\
 & 200 \\
 & \underline{200} \\
 & \underline{0}
\end{array}
$$

Hence, the decimal expansion of $\frac{1237}{25}$ is 49.48. Since the remainder is obtained as zero, the decimal number is terminating.

- If x is a rational number with terminating decimal expansion then it can be expressed in the $\frac{p}{q}$ form, where p and q are co-prime (the HCF of p and q is 1) and the prime factorisation of q is of the form 2^n5^m, where n and m are non-negative integers.
- Let $x = \frac{p}{q}$ be any rational number.

i. If the prime factorization of q is of the form 2^m5^n, where m and n are non-negative integers, then x has a terminating decimal expansion.

ii. If the prime factorisation of q is not of the form 2^m5^n, where m and n are non-negative integers, then x has a non-terminating and repetitive decimal expansion.

For example, $\frac{17}{1600} = \frac{17}{2^6 \times 5^2}$ has the denominator in the form 2^n5^m, where $n = 6$and $m = 2$ are non-negative integers. So, it has a terminating decimal expansion.

$\frac{723}{392} = \frac{3 \times 241}{2^3 \times 7^2}$ has the denominator not in the form 2^n5^m, where n and m are non-negative integers. So, it has a non-terminating decimal expansion.