Triangles

- **Similar and Congruent Figures**
  - Two geometric figures having the same shape and size are said to be congruent figures.
  - Two geometric figures having the same shape, but not necessarily the same size, are called similar figures.

    Example:
    (1) All circles are similar.
    (2) All equilateral triangles are similar.
    (3) All congruent figures are similar. However, the converse is not true.

- **Similarity of Polygons**
  Two polygons with the same number of sides are similar, if
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  - their corresponding angles are equal
  - their corresponding sides are in the same ratio (or proportion)

- Two lines segments are congruent, if they are equal in length.
- Two angles are congruent, if they have the same measure.
- **CPCT:**

  CPCT stands for Corresponding Parts of Congruent Triangles.
  If ΔABC ≅ ΔPQR, then corresponding sides are equal i.e., AB = PQ, BC = QR, and CA = RP and corresponding angles are equal i.e., ∠A = ∠P, ∠B = ∠Q, and ∠C = ∠R.

- **Basic proportionality theorem:**
  If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.

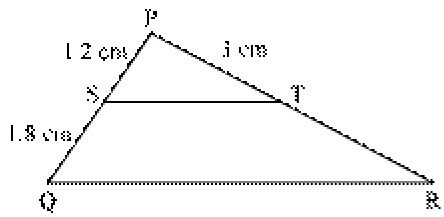
  Corollary: If D and E are points on the sides, AB and AC, respectively of ΔABC such that DE || BC, then

  $$\frac{AB}{AD} = \frac{AC}{AE}$$

  $$\frac{AB}{DB} = \frac{AC}{EC}$$

  Example:

  In the given figure, S and T are points on PQ and PR respectively of ΔPQR such that ST || QR.



Determine the length of PR.

Solution:
Since ST || QR, by basic proportionality theorem, we have

$$\frac{PS}{SQ} = \frac{PT}{TR}$$

$$\Rightarrow \frac{1.2}{1.8} = \frac{3}{TR}$$

$$\Rightarrow TR = \frac{3 \times 1.8}{1.2} = 4.5 \text{ cm}$$

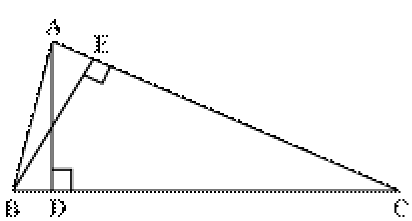
$$\therefore PR = PT + TR = (3 + 4.5) \text{ cm} = 7.5 \text{ cm}$$

- **Converse of basic proportionality theorem:**
  If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.
- **Theorem: (AAA similarity criterion)**
  If in two triangles, corresponding angles are equal, then their corresponding sides are in the same ratio (or proportion) and hence, the two triangles are similar.

- **Theorem: (AA similarity criterion)**
  If in two triangles, two angles of one triangle are respectively equal to the two angles of the other triangle, then the two triangles are similar.

Example:
In ΔABC, ∠C is acute, D and E are points on sides BC and AC respectively, such that AD⊥BC and BE⊥AC. Show that BC × CD = AC × CE.

Solution:



In ΔADC and ΔBEC,
∠ADC = ∠BEC = 90°

∠DCA = ∠ECB [Common]

∴ By AA similarity criterion, ΔADC ~ ΔBEC

$$\therefore \frac{CD}{CE} = \frac{AC}{BC}$$

$$\Rightarrow BC \times CD = AC \times CE$$

Hence, the result is proved.

- **Theorem: (SSS similarity criterion)**

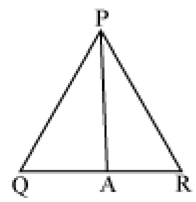
  If in two triangles, sides of one triangle are proportional to the sides of the other triangle then the two triangles are similar by SSS similarity criterion.

  Example:

  If PQR is an isosceles triangle with PQ = PR and A is the mid-point of side QR then prove that ΔPAQ is similar to ΔPAR.

  Solution:

  It is given that ΔPQR is an isosceles triangle and PQ = PR.



  In triangles PAQ and PAR,

  PQ = PR

  Also, A is the mid-point of QR, therefore QA = AR

  And, PA = PA (Common to both triangles)

  Therefore, we can say that

$$\frac{PQ}{PR} = \frac{QA}{AR} = \frac{PA}{PA}$$

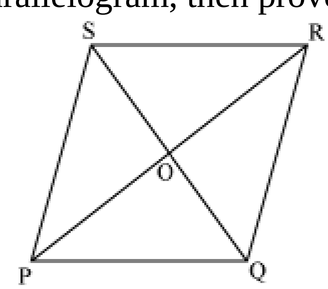
  ∴ Using SSS similarity criterion, we obtain ΔPAQ ~ ΔPAR

- **Theorem: (SAS similarity criterion)**

  If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar by SAS similarity criterion.

  Example:

  If PQRS is a parallelogram, then prove that ΔSOR is similar to ΔPOQ.



  Solution:

  Consider ΔSOR and ΔPOQ.

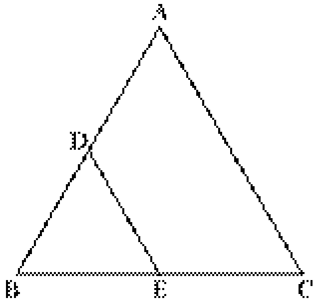
Since PQRS is a parallelogram, the diagonals bisect each other.
∴ SO = OQ and PO = OR
and ∠POQ = ∠SOR (Vertically opposite angles)
By SAS similarity criterion, we obtain
ΔSOR ~ ΔQOP

- **Areas of similar triangles**

Theorem: The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Example:
In ΔABC, D and E are the respective mid-points of sides AB and BC. Find the ratio of the areas of ΔDBE and ΔABC.

Solution:



In ΔABC, D and E are the respective mid-points of the sides, AB and BC.
By the converse of BPT, DE||AC
In ΔDBE and ΔABC,
∠DBE = ∠ABC [Common]
∠BED = ∠BCA [Corresponding angles]
∠BDE = ∠BAC [Corresponding angles]
∴ By AAA similarity criterion, ΔDBE~ ΔABC

$$\Rightarrow \frac{\text{Area}(\Delta DBE)}{\text{Area}(\Delta ABC)} = \left(\frac{BE}{BC}\right)^2$$

$$= \left(\frac{BE}{2BE}\right)^2 \quad [\text{E is mid-point of BC}]$$

$$= \frac{1}{4}$$

$$= 1:4$$

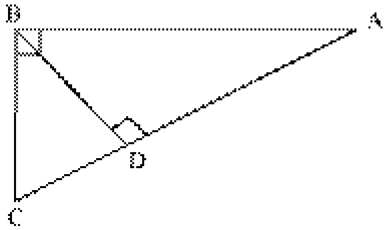
Result: Using the above theorem, the result " the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians or altitudes or angle bisector" can be proved.

- **Pythagoras theorem:**

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Example:

ΔABC is right-angled at B and BD⊥CA.



Prove that $BD^2 = CD \times DA$.

Solution:

By applying Pythagoras theorem in ΔBDC, ΔBDA, and ΔABC, we obtain

$BC^2 = CD^2 + BD^2$ … (1)

$BA^2 = BD^2 + DA^2$ … (2)

$CA^2 = BC^2 + BA^2$ … (3)

Adding equations (1) and (2), we obtain

$BC^2 + BA^2 = 2BD^2 + CD^2 + DA^2$

$\Rightarrow CA^2 = 2BD^2 + CD^2 + DA^2$ ... [Using (3)]

$\Rightarrow (CD + DA)^2 = 2BD^2 + CD^2 + DA^2$

$\Rightarrow CD^2 + DA^2 + 2 \times CD \times DA = 2BD^2 + CD^2 + DA^2$

$\Rightarrow CD \times DA = BD^2$

- **Converse of Pythagoras theorem:**

  In a triangle, if the square of one side is equal to the sum of the squares of other two sides, then the angle opposite to the first side is a right angle.