CHAPTER

6

APPLICATIONS OF DERIVATIVES

Syllabus

Applications of derivatives: rate of change of bodies, increasing/decreasing functions, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

In this chapter you will study

- *Rate of change of bodies*
- *increasing and decreasing functions, tangent and normals*
- *maxima and minima*

List of Topics

Topic-1 Rate of Change of Bodies

Revision Notes

Interpretation of $\frac{dy}{dx}$ as a rate measure:

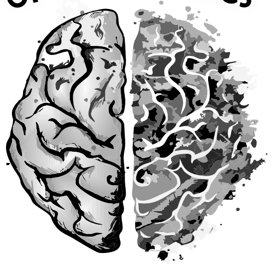
If two variables x and y are varying with respect to another variables say t, *i.e.*, if $x = f(t)$, then by the Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Also, if y is a function of x and they are related as $y = f(x)$ then, $f'(\alpha)$, *i.e.*, represents the rate of change of y with respect to x at the instant when $x = \alpha$.

Applications of Derivatives



Rate of change of bodies

The change in quantity w.r.t. time is known as rate of change, If a quantity y varies w.r.t. another quantity x, satisfying $y = f(x)$, then $\frac{dy}{dx}$ represents rate of change of y w.r.t. 'x'

Increasing and decreasing functions

If $f'(x) \geq 0 \; \forall \; x \in (a,b)$ then f is increasing in (a,b) and if $f'(x) \leq 0 \; \forall \; x \in (a,b)$, then f is decreasing in (a,b)

eg: Let $f(x) = x^3 - 3x^2 + 4x, \; x \in R$, then

$f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \; \forall \; x \in R$.

So, the function f is strictly increasing on R.

A function f is said to be

(i) increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \leq f(x_2) \; \forall \; x_1, x_2 \in (a,b)$, and

(ii) decreasing on (a, b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \geq f(x_2) \; \forall \; x_1, x_2 \in (a,b)$

Maxima and Minima

A point C in the domain of 'f' at which either $f'(C) = 0$ or is not differentiable is called a critical point of f.

First derivative test

Let f be continuous at a critical point C in open interval. Then

(i) if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima.

(ii) If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.

(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second derivative test

Let f be a function defined on given interval, f is twice differentiable at C. Then

(i) $x = C$ is a point of local maxima If $f'(C) = 0$ and $f''(C) < 0$, $f(C)$ is local maxima of f.

(ii) $x = C$ is a point of local minima if $f'(C) = 0$ and $f''(C) > 0$. $f(C)$ is local minima of f.

(iii) The test fails if $f'(C) = 0$ and $f''(C) = 0$

Trace the Mind Map

- First Level
- Second Level
- Third Level

OBJECTIVE TYPE QUESTIONS

Multiple Choice Questions

Q. 1. If the rate of change of radius of a circle is 6 cm/s, then find the rate of change of area of the circle, when $r = 3$.

(A) 24π cm²/s **(B)** 36π cm²/s
(C) 12π cm²/s **(D)** 21π cm²/s

Ans. Option (B) is correct.

Explanation : The rate of change of radius of circle is $\frac{dr}{dt} = 6$ cm/s

$$A = \pi r^2$$

Differentiating w.r.t. 't', we get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r(6)$$

$$\Rightarrow \quad \frac{dA}{dt} = 12\pi r$$

$$\therefore \quad \left(\frac{dA}{dt}\right)_{\text{at } r=3} = 12\pi \times 3 = 36\pi \text{ cm}^2/\text{s}$$

Q. 2. The edge of a cube is increasing at a rate of 7 cm/s. Find the rate of change of area of the cube when a = 3 cm.

(A) 252 cm²/s **(B)** 504 cm²/s
(C) 498 cm²/s **(D)** 287 cm²/s

Ans. Option (A) is correct.

Explanation: Let the edge of the cube be a.
The rate of change of edge of the cube is given by
The area of the cube is $A = 6a^2$
Differentiating w.r.t. 't', we get

$$\frac{dA}{dt} = 12a\frac{da}{dt}$$

$$\therefore \quad \frac{dA}{dt} = 12a \times 7 = 84a$$

$$\text{Thus,} \quad \left.\frac{dA}{dt}\right|_{a=3} = 84 \times 3 = 252 \text{ cm}^2/\text{s}$$

Q. 3. The rate of change of area of square is 40 cm²/s. What will be the rate of change of side if the side is 5 cm.

(A) 2 cm/s **(B)** 4 cm/s
(C) 6 cm/s **(D)** 8 cm/s

Ans. Option (B) is correct.

Explanation: Let the side of the square be x.
Area of square, $A = x^2$
Differentiating w.r.t. 't', we get

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

$$\text{Since, given} \quad \frac{dA}{dt} = 40 \text{ cm}^2/\text{s}$$

$$\therefore \quad 40 = 2x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{20}{x}$$

$$\text{Thus,} \quad \left.\frac{dx}{dt}\right|_{x=5} = \frac{20}{5} = 4 \text{ cm/s}$$

Q. 4. At what rate will the lateral surface area of the cylinder increase if the radius is increasing at the rate of 2 cm/s when the radius is 5 cm and height is 8 cm?

(A) 25π cm/s **(B)** 30π cm/s
(C) 32π cm/s **(D)** 40π cm/s

Ans. Option (C) is correct.

Explanation: Let r be the radius and h be the height of the cylinder.

$$\text{Then,} \quad \frac{dr}{dt} = 2 \text{ cm/sec} \qquad \text{(given)}$$

The area of the cylinder is given by $A = 2\pi rh$
Differentiating w.r.t. 't', we get

$$\frac{dA}{dt} = 2\pi\frac{dr}{dt}\times h$$

$$\therefore \quad \frac{dA}{dt} = 2\pi(2) \times h = 4\pi h$$

$$\text{Now,} \quad \left.\frac{dA}{dt}\right|_{h=8} = 4\pi(8) = 32\pi \text{ cm/s}$$

Q. 5. If the total cost P(x) in ₹ associated with an item is given by $P(x) = 0.4x^2 + 2x - 10$, then the marginal cost if no. of units produced is 10 is :

(A) ₹8 **(B)** ₹10
(C) ₹2 **(D)** ₹7

Ans. Option (B) is correct.

Explanation : The marginal cost is the rate of change of cost w.r.t. the no. of units produced.

i.e., $$\text{Marginal cost (MC)} = \frac{dP(x)}{dt}$$

$$= \frac{d}{dt}(0.4x^2 + 2x - 10)$$

$$= 0.8x + 2$$

$$\therefore \quad \text{Marginal cost (MC)}\big|_{\text{at } x=8} = 0.8 \times 10 + 2 = ₹10$$

Q. 6. If the circumference of the circle is changing at the rate of 10 cm/s, then at what rate, the area of the circle changes, if radius is 3 cm?

(A) 20 cm²/s **(B)** 40 cm²/s
(C) 30 cm²/s **(D)** 50 cm²/s

Ans. Option (C) is correct.

Explanation : Circumference of circle with radius r is given by $C = 2\pi r$
Differentiating w.r.t. 't', we get

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

Given, $\frac{dC}{dt} = 10$ cm/s

$\therefore \quad 10 = 2\pi\frac{dr}{dt}$

$\Rightarrow \quad \frac{dr}{dt} = \frac{5}{\pi}$ cm/s

Now, Area of circle, $A = \pi r^2$

$\therefore \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

Substituting $r = 3$ cm and $\frac{dr}{dt} = \frac{5}{\pi}$ cm/s, we get

$$\frac{dA}{dt} = 2\pi \times 3 \times \frac{5}{\pi}$$

$\therefore \quad \frac{dA}{dt} = 30$ cm²/s

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. The radius of a circle is increasing at the uniform rate of 3 cm/s. At the instant the radius of the circle is 2 cm, then at what rate area increases. A I U [CBSE Board 2020]

Topper Answer, 2020

Sol. $\frac{dr}{dt} = 3$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

$\left(\frac{dA}{dt}\right)_{r=2} = 2\pi(2)(3) = 12\pi$ cm²/sec **1**

Q. 2. The total cost associated with provision of free mid-day meals to x students of a school in primary classes is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$. If the marginal cost is given by rate of change $\frac{dC}{dx}$ total cost, write the marginal cost of food for 300 students. U

Sol. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$

$C'(x) = 0.015x^2 - 0.04x + 30 + 0$

$\Rightarrow \quad C'(300) = 0.015(300)^2 - 0.04(300) + 30$

$= 1368$

So the marginal cost of food for 300 students is ₹1,368. **1**

Q. 3. The total expenditure (in ₹) required for providing the cheap edition of a book for poor and deserving students is given $R(x) = 3x^2 + 36x$, where x is the number of set of books. If the marginal expenditure is defined as $\frac{dR}{dx}$, write the marginal expenditure required for 1200 such sets. U

Q. 4. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $p(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added. A I

Sol. Here, pollution content is given by $p(x) = 0.005x^3 + 0.02x^2 + 30x$, where x is the number of diesel vehicles

$\therefore \quad \frac{dp}{dx} = 0.015x^2 + 0.04x + 30$

$$\left(\frac{dp}{dx}\right)_{x=3} = 0.015(3)^2 + 0.04(3) + (30)$$

$= 30.255$ **1**

Q. 5. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in ₹) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$. U

Sol. Total revenue is given by $R(x) = 3x^2 + 36x + 5$

Marginal revenue $\frac{dR}{dx} = 6x + 36$

At $x = 5$, $\left(\frac{dR}{dx}\right)_{x=5} = 6 \times 5 + 36 = 66$ **1**

Q. 6. If $C = 0.003x^3 + 0.02x^2 + 6x + 250$ gives the amount of carbon pollution in air in an area on the entry of x number of vehicles, then find the marginal carbon pollution in the air, when 3 vehicles have entered in the area. U

Sol. Given, $C = 0.003x^3 + 0.02x^2 + 6x + 250$

$$\frac{dC}{dx} = 0.009x^2 + 0.04x + 6$$

$$\left(\frac{dC}{dx}\right)_{x=3} = 0.009(3)^2 + 0.04(3) + 6$$

$$= 0.081 + 0.12 + 6$$

$$= 6.201$$ **1**

Q. 7. The contentment obtained after eating x-units of a new dish at a trial function is given by the function $f(x) = x^3 + 6x^2 + 5x + 3$. If the marginal contentment is defined as the rate of change of $f(x)$ with respect to the number of units consumed at an instant, then find the marginal contentment when three units of dish are consumed. U

Short Answer Type Questions-I (2 marks each)

Q. 1. The length x of a rectangle is decreasing at the rate of 5 cm/min and the width y increasing at the rate of 4 cm/min, find the rate of change its area when $x = 5$ cm and $y = 8$ cm. A I

Sol. Let A denote the area of rectangle at instant t.

$\therefore$ $A = xy$ (area of rectangle),

$\Rightarrow$ $\frac{dx}{dt} = -5$ cm/min, $\frac{dy}{dt} = 4$ cm/min **(given)**

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$ **1**

$$\Rightarrow \left(\frac{dA}{dt}\right)_{x=5,\, y=8} = 5 \times 4 + 8 \times (-5) \text{ cm}^2\text{/min}$$

$$\Rightarrow \frac{dA}{dt} = (20 - 40) \text{ cm}^2\text{/min}$$

$$\Rightarrow \frac{dA}{dt} = -20 \text{ cm}^2\text{/min}$$ **1**

Here, (–) ve sign shows that area is decreasing at the rate of 20 cm²/min.

Q. 2. The volume of a cube is increasing at the rate of 9 cm³/sec. How fast is the surface area increasing when the length of an edge is 10 cm? U

Q. 3. Find the point on the curve $y^2 = 8x + 3$ for which the y-coordinate change 8 times more than coordinate of x. A I

Sol. $y^2 = 8x + 3$ ...(i) (given)

$$\therefore 2y\frac{dy}{dt} = 8\frac{dx}{dt}$$

$$\frac{dy}{dt} = 8\frac{dx}{dt}$$ ...(ii) (given)

$$\therefore 2y8\frac{dx}{dt} = 8\frac{dx}{dt}$$ **1**

$$\Rightarrow y = \frac{8}{16} = \frac{1}{2}$$

For $y = \frac{1}{2}$

From eq (i), $\left(\frac{1}{2}\right)^2 = 8x + 3$

or, $\frac{1}{4} - 3 = 8x$

or, $x = -\frac{11}{32}$

Hence, required point is $\left(-\frac{11}{32}, \frac{1}{2}\right)$. **1**

Q. 4. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area increases, when the side is 10 cm. U

Sol. Let S be the side of an equilateral triangle,

then $\frac{dS}{dt} = 2$ cm/sec

and Area (A) $= \frac{\sqrt{3}S^2}{4}$ **1**

$$\therefore \frac{dA}{dt} = \frac{2\sqrt{3}S}{4}.\frac{dS}{dt}$$

$$= \frac{2\sqrt{3}}{4}.20 \quad \text{(when S = 10 cm)}$$

$$= 10\sqrt{3} \text{ cm}^2\text{/s}$$ **1**

Short Answer Type Questions-II (3 marks each)

Q. 1. Volume of the cube is increasing at a rate of 9 cubic inches per second. What is the rate at which surface area is increasing when the length of the edge of the cube is 12 inches? A I

Sol. Given, increasing rate of volume of cube, $\frac{dV}{dt} = 9$ cubic inches/s

Since, Volume of a cube, $V = a^3$ **1**

Differentiating both sides w.r.t 't', we get

$$\frac{dV}{dt} = 3a^2\frac{da}{dt}$$

$$\therefore \quad 9 = 3a^2\frac{da}{dt}$$

$$\Rightarrow \quad \frac{da}{dt} = \frac{3}{a^2} \quad \text{...(i) } \mathbf{1}$$

Now, surface area of cube, S = $6a^2$

$\therefore$ Rate of change of surface area, $\frac{dS}{dt} = 12a\frac{da}{dt}$

$$\therefore \quad \frac{dS}{dt} = 12.a\frac{3}{a^2} \quad \text{[from (i)]}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{36}{12} = 3 \text{ inches/s}$$

[Given a = 12 inches] **1**

Q. 2. The surface area of a spherical balloon is increasing at the rate of 2 cm²/s. At what rate is the volume of the balloon is increasing when the radius of the balloon is 8 cm ? U

Sol. Let r be the radius, A be the surface area and V be the volume of the spherical balloon at time t seconds.

Then, $A = 4\pi r^2$

$$\therefore \quad \frac{dA}{dt} = 8\pi r\frac{dr}{dt}$$

(Differentiating both sides w.r.t. 't')

Given, $\frac{dA}{dt} = 2 \text{ cm}^2/\text{s}$

$$\therefore \quad 2 = 8\pi r\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{2}{8\pi r} = \frac{1}{4\pi r} \quad \text{...(i) } \mathbf{1}$$

Now, volume of spherical bottom V = $\frac{4}{3}\pi r^3$

$$\therefore \quad \frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

(Differentiating both sides w.r.t. 't')

$$\text{or,} \quad \frac{dV}{dt} = 4\pi r^2 \times \frac{1}{4\pi r} \quad \text{[from eq (i)]}$$

$$\text{or,} \quad \frac{dV}{dt} = r$$

$$\text{or,} \quad \frac{dV}{dt} = 8 \text{ cm}^3/\text{s} \quad [\text{Given } r = 8 \text{ cm}]$$

Therefore, the volume of the balloon is increasing at the rate of 8 cm³/s.

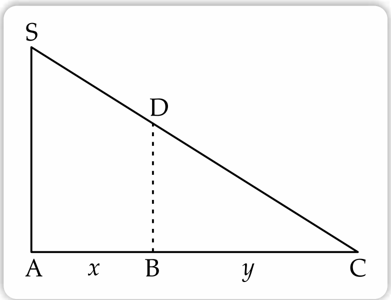
Q. 3. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing, when the foot of the ladder is 4 m away the wall? U

Long Answer Type Questions (5 marks each)

Q. 1. Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm²/sec in the surface, through a tiny hole at the vertex of bottom. When the slant height of water level is 4 cm, find the rate of decrease of the slant height of the water. A I

Q. 2. A point source of light is hung 40 feet directly above a straight horizontal path on which a man of 5 feet in height is walking. How fast will the man's shadow lengthen and how fast will the tip of shadow move when he is walking away from the light at the rate of 140 ft/min. U

Sol. Let S be the position of source of light.



Let BD be the position of the man at time t.

Let AB = x and BC = length of the shadow = y

$$\text{Now,} \quad \frac{dx}{dt} = 140 \quad \mathbf{1}$$

Here, $\Delta ASC \sim \Delta BDC$

$$\therefore \quad \frac{AS}{BD} = \frac{AC}{BC}$$

$$\Rightarrow \quad \frac{40}{5} = \frac{x+y}{y}$$

$$\Rightarrow \quad 8y = x + y$$

$$\Rightarrow \quad 7y = x \quad \mathbf{1}$$

Differentiating both sides w.r.t. 't', we get

$$7\frac{dy}{dt} = \frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{1}{7}(140) = 20 \text{ ft/min} \quad \mathbf{1}$$

Therefore, the shadow of the man is lengthening at the rate of 20 ft/min.

The tip of the shadow is at C. Let AC = Z

$$\text{Also,} \quad \frac{AS}{BD} = \frac{AC}{BC} \quad [\because \Delta ASC \sim \Delta BDC]$$

$$\therefore \quad \frac{40}{5} = \frac{z}{z-x}$$

$$\Rightarrow \quad 8(z-x) = z$$

$$\Rightarrow \quad 7z = 8x \quad \mathbf{1}$$

Differentiating both sides w.r.t. 't', we get

$$7\frac{dz}{dt} = 8\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{8}{7} \times 140$$

$$= 160 \text{ ft/min} \quad \mathbf{1}$$

Therefore, the tip of the shadow is moving at the rate 160 ft/min.

Topic-2 Increasing/Decreasing Functions

Concepts Covered • *Increasing function,* • *Decreasing function,* • *Constant function* • *Monotonic function*

Revision Notes

1. A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta$, $f(\alpha) > f(\beta)$.

If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

2. A function $f(x)$ is said to be a **decreasing function** in $[a, b]$, if, as x increases, $f(x)$ decreases *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.

If $f'(x) \leq 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$ provided $f(x)$ is continuous at $x = a$ and $x = b$.

➲ A function $f(x)$ is a **constant function** in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

➲ By **monotonic function** $f(x)$ in interval I, we mean that f is either **only increasing** in I or **only decreasing** in I.

3. Finding the intervals of increasing and/or decreasing of a function:

ALGORITHM

STEP 1: Consider the function $y = f(x)$.

STEP 2: Find $f'(x)$.

STEP 3: Put $f'(x) = 0$ and solve to get the critical point(s).

STEP 4: The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing; and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

OBJECTIVE TYPE QUESTIONS

Multiple Choice Questions

Q. 1. The function $y = x^2e^{-x}$ is decreasing in the interval

(A) $(0, 2)$ **(B)** $(2, \infty)$
(C) $(-\infty, 0)$ **(D)** $(-\infty, 0) \cup (2, \infty)$

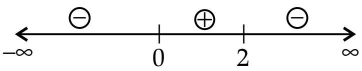
[CBSE Board 2021]

Ans. Option (D) is correct.

Explanation: We have,

$$f(x) = y = x^2e^{-x}$$

$$\therefore \frac{dy}{dx} = 2x\,e^{-x} + x^2(-1)e^{-x} = xe^{-x}(2 - x)$$



Now, put $\frac{dy}{dx} = 0$

$\Rightarrow x = 0$ and $x = 2$

The points $x = 0$ and $x = 2$ divide the real line into three disjoint intervals *i.e.*, $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$.

In intervals, $(-\infty, 0)$ and $(2, \infty)$, $f'(x) < 0$ as e^{-x} is always positive.

$\therefore f(x)$ or y is decreasing in $(-\infty, 0)$ and $(2, \infty)$.

Q. 2. The function $f(x) = 2x^3 - 15x^2 + 36x + 6$ is increasing in the interval

(A) $(-\infty, 2) \cup (3, \infty)$ **(B)** $(-\infty, 2)$
(C) $(-\infty, 2] \cup [3, \infty)$ **(D)** $(3, \infty)$

[CBSE Board 2021]

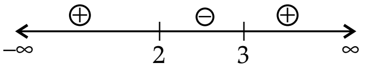
Ans. Option (C) is correct.

Explanation: Given, $f(x) = 2x^3 - 15x^2 + 36x + 6$

$\therefore$ $f'(x) = 6x^2 - 30x + 36$

It $f'(x) \geq 0$, then $f(x)$ is increasing.

So, $6x^2 - 30x + 36 \geq 0$



or, $x^2 - 5x + 6 > 0$

or, $(x - 3)(x - 2) > 0$

$\therefore$ $x \in (-\infty, 2] \cup [3, \infty)$

Q. 3. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is:

(A) $[-1, \infty)$ **(B)** $[-2, -1]$
(C) $(-\infty, -2]$ **(D)** $[-1, 1]$

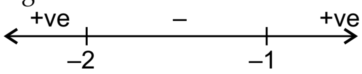
Ans. Option (B) is correct.

Explanation: Given that,

$$f(x) = 2x^3 + 9x^2 + 12x - 1$$
$$f'(x) = 6x^2 + 18x + 12$$
$$= 6(x^2 + 3x + 2)$$
$$= 6(x + 2)(x + 1)$$

So, $f'(x) < 0$, for decreasing.

On drawing number lines as below:



We see that $f'(x)$ is decreasing in $[-2, -1]$.

Q. 4. $y = x(x-3)^2$ **decreases for the values of** x **given by:**

(A) $1 < x < 3$ **(B)** $x < 0$

(C) $x > 0$ **(D)** $0 < x < \frac{3}{2}$

Ans. Option (A) is correct.

***Explanation*:** Given that,

$$y = x(x-3)^2$$

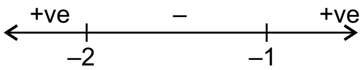
$$\therefore \quad \frac{dy}{dx} = x.2(x-3).1 + (x-3)^2.1$$

$$= 2x^2 - 6x + x^2 + 9 - 6x$$

$$= 3x^2 - 12x + 9$$

$$= 3(x^2 - 3x - x + 3)$$

$$= 3(x-3)(x-1)$$



So, $y = x(x-3)^2$ decreases for (1, 3).

[Since, $y' < 0$ for all $x \in (1, 3)$, hence y is decreasing on (1, 3)]

Q. 5. The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ **is strictly**

(A) increasing in $\left(\pi, \frac{3\pi}{2}\right)$

(B) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(C) decreasing in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(D) decreasing in $\left(0, \frac{\pi}{2}\right)$

Ans. Option (B) is correct.

***Explanation*:** Given that,

$$f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$$

On differentiating with respect to x, we get

$$f'(x) = 12\sin^2 x.\cos x - 12\sin x.\cos x + 12\cos x$$

$$= 12\cos x[\sin^2 x - \sin x + 1]$$

$$\Rightarrow f'(x) = 12\cos x[\sin^2 x + 1(1 - \sin x)]$$

$$\Rightarrow 1 - \sin x \geq 0 \text{ and } \sin^2 x \geq 0$$

$$\Rightarrow \sin^2 x + 1 - \sin x \geq 0$$

Hence, $f'(x) > 0$, when $\cos x > 0$, *i.e.*, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

So, $f(x)$ is increasing when $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

and $f'(x) < 0$, when $\cos x < 0$, *i.e.*, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Hence, $f'(x)$ is decreasing when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Since $\left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$

Q. 6. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$.

(A) $\sin 2x$ **(B)** $\tan x$

(C) $\cos x$ **(D)** $\cos 3x$

Ans. Option (C) is correct.

***Explanation*:** In the given interval $\left(0, \frac{\pi}{2}\right)$

$$f(x) = \cos x$$

On differentiating with respect to x, we get

$$f'(x) = -\sin x$$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

Q. 7. The function $f(x) = \tan x - x$

(A) always increases

(B) always decreases

(C) never increases

(D) sometimes increases and sometimes decreases

Ans. Option (A) is correct.

***Explanation*:** We have,

$$f(x) = \tan x - x$$

On differentiating with respect to x, we get

$$f'(x) = \sec^2 x - 1$$

$$f'(x) = \tan^2 x$$

$$\Rightarrow \quad f'(x) > 0, \forall x \in R$$

So, $f(x)$ always increases.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Show that the function given by $f(x) = \sin x$ **is strictly decreasing in** $\left(\frac{\pi}{2}, \pi\right)$.

Sol. Consider, $f(x) = \sin x$

$$f'(x) = \cos x \quad \text{.......(i)}$$

$\cos x < 0$ for each $x \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore$ $f(x) < 0$ [from (i)]

Hence, function is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$. **1**

Q. 2. Show that function $y = 4x - 9$ is increasing for all $x \in R$.

Q. 3. Find the interval for which the function $f(x) = \cot^{-1} x + x$ increases.

Sol. Given, $f(x) = \cot^{-1}x + x$

$$f'(x) = \frac{-1}{1+x^2} + 1 = \frac{-1+1+x^2}{1+x^2}$$

$$= \frac{x^2}{1+x^2} \geq 0 \text{ for all } x \in \text{R}.$$

$\therefore$ f increases for $(-\infty, \infty)$. **1**

Short Answer Type Questions-I (2 marks each)

Q. 1. Show that the function $f(x) = \frac{x}{3} + \frac{3}{x}$ decreases in the intervals $(-3, 0) \cup (0, 3)$.

R&U [CBSE OD Set-III 2020]

Sol. $f'(x) = \frac{1}{3} - \frac{3}{x^2}$ ½

for decreasing, $f'(x) < 0 \Rightarrow \frac{1}{3} - \frac{3}{x^2} < 0$ ½

$\Rightarrow x^2 < 9 \Rightarrow -3 < x < 3$

since $f(x)$ is not defined at $x = 0$.

so $f(x)$ decreasing in $(-3, 0) \cup (0, 3)$. **1**

[CBSE Marking Scheme 2020]

Detailed Solution:

$$f(x) = \frac{x}{3} + \frac{3}{x}$$

$$f'(x) = \frac{1}{3} - \frac{3}{x^2}$$

$$f'(x) = 0$$

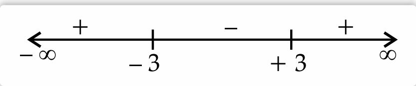
$$\frac{1}{3} - \frac{3}{x^2} = 0$$

$$\frac{x^2 - 9}{3x^2} = 0$$

$x^2 - 9 = 0$, x not equal to 0

$(x + 3)(x - 3) = 0$

$x = -3, 3$



Here $x \neq 0$

The function decreases in the interval $(-3, 0) \cup (0, 3)$

Commonly Made Error

- Students add the function and apply quotient rule for finding the derivative which consumes time.

Answering Tips

- Practice more problems on finding increasing/decreasing intervals.

Q. 2. Show that the function f defined by $f(x) = (x - 1)e^x + 1$ is an increasing function for all $x > 0$.

R&U [CBSE OD Set-I 2020]

Q. 3. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R.

R&U [NCERT] [O.D. Set I 2017]

Sol. $f(x) = x^3 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$ ½

$= 3[x^2 - 2x + 2] = 3[(x - 1)^2 + 1]$ **1**

since $f'(x) > 0$; $x \in R$

$\therefore$ $f(x)$ is increasing on R. ½

[CBSE Marking Scheme, 2017]

Detailed Solution:

Topper Answer, 2017

Sol.

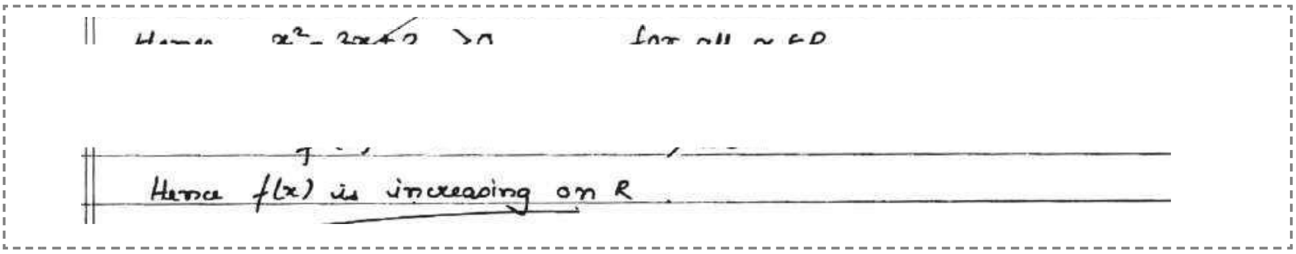
$f(x) = x^3 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$

$= 3(x^2 - 2x + 2)$

discriminant of the formed quadratic $= b^2 - 4ac = (-2)^2 - 4(1)(2)$

$= 4 - 8 = -4$


Hence f(x) is increasing on R.

Q. 4. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing or R. R&U [Delhi 2017]

Q. 5. Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$ is decreasing for all $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. R&U [Foreign 2017]

Sol. $$f'(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}$$ **1**

$1 + (\sin x + \cos x)^2 > 0;\ x \in R$

and $\frac{\pi}{4} < x < \frac{\pi}{2}$ or $\cos x < \sin x$ or $\cos x - \sin x < 0$ **½**

or $f'(x) < 0$ or $f(x)$ is decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. **½**

[CBSE Marking Scheme 2017]

Short Answer Type Questions-II (3 marks each)

Q. 1. Find the intervals in which the function f given by $f(x) = \tan x - 4x,\ x \in \left(0, \frac{\pi}{2}\right)$ is

(a) strictly increasing (b) strictly decreasing

A I R&U [CBSE SQP 2020-21]

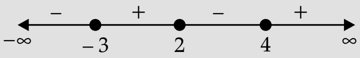
Q. 2. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is (a) strictly increasing, (b) strictly decreasing. R&U [Delhi/OD 2018]

Sol. $f'(x) = x^3 - 3x^2 - 10x + 24$

$= (x - 2)(x - 4)(x + 3)$ **1**

$f'(x) = 0 \Rightarrow x = -3, 2, 4.$

sign of $f'(x)$:



$\therefore f(x)$ is strictly increasing on $(-3, 2) \cup (4, \infty)$ **1**

and $f(x)$ is strictly decreasing on $(-\infty, -3) \cup (2, 4)$ **1**

[CBSE Marking Scheme, 2018] (Modified)

Q. 3. Find the intervals in which the function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is (i) Strictly increasing (ii) Strictly decreasing R&U [O.D Comptt Set I, II, III 2018]

Sol. $f(x) = -2x^3 - 9x^2 - 12x + 1$

Now, $f'(x) = -6x^2 - 18x - 12$

$= -6[x^2 + 3x + 2]$

$= -6[x^2 + 2x + x + 2]$

$f'(x) = -6(x + 1)(x + 2)$ **1**

Put, $f'(x) = 0 \Rightarrow x = -2, x = -1$

$\Rightarrow$ Intervals are $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$

Getting $f'(x) > 0$ in $(-2, -1)$ and $f'(x) < 0$ in $(-\infty, -2) \cup (-1, \infty)$ **1**

$\Rightarrow f(x)$ is strictly increasing in $(-2, -1)$

and strictly decreasing in $(-\infty, -2) \cup (-1, \infty)$ **1**

[CBSE Marking Scheme, 2018] (Modified)

Q. 4. Find the intervals in which the function given by $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$ is

(i) increasing

(ii) decreasing. R&U [NCERT] [Delhi Comptt., 2017]

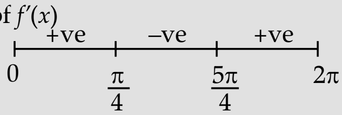
Sol. $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$ **1**

$f'(x) = 0$ or $\cos x = \sin x$ **1**

$\therefore\ x = \frac{\pi}{4}, \frac{5\pi}{4}$ **1**

Sign of $f'(x)$



So $f(x)$ is strictly increasing in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

and strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ **1**

[CBSE Marking Scheme, 2017] (Modified)

Detailed Solution:

$f(x) = \sin x + \cos x$

or $f'(x) = \cos x - \sin x$

Now, $f'(x) = 0$ gives $\sin x = \cos x$ which gives

$x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \le x \le 2\pi$

$\tan x = 1$

The points $x = \frac{\pi}{4}, \frac{5\pi}{4}$ divides the interval $[0, 2\pi]$ into 3 disjoint intervals, $\left[0, \frac{\pi}{4}\right)$, $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$.

Note that $f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ or $f(x)$ is strictly increasing in intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$.

Also $f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

or $f(x)$ is strictly decreasing in this interval.

Interval	Sign of $f'(x)$	Nature of function
$\left[0, \frac{\pi}{4}\right)$	+ ve	f is strictly increasing
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	– ve	f is strictly decreasing
$\left(\frac{5\pi}{4}, 2\pi\right]$	+ ve	f is strictly increasing

1

Q. 5. Determine for what values of x, the function $f(x) = x^3 + \frac{1}{x^3}$ $(x \neq 0)$ is strictly increasing or strictly decreasing.

A [SQP Dec. 2016-17] [NCERT][HOTS]

Q. 6. Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ on $\left[0, \frac{\pi}{2}\right]$. R&U [Outside Delhi 2016]

[NCERT] [HOTS]

Sol. Getting $\frac{dy}{d\theta} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$ **1**

Equating $\frac{dy}{d\theta}$ to 0 and getting critical point as $\cos\theta = 0$ *i.e.*, $\theta = \frac{\pi}{2}$

For all θ, $0 \le \theta \le \frac{\pi}{2}$, $\frac{dy}{d\theta} \ge 0$ **1**

Hence, y is an increasing function of θ on $\left[0, \frac{\pi}{2}\right]$. **1**

[CBSE Marking Scheme, 2016] (Modified)

Commonly Made Error

- Sometimes candidates do not have any basic knowledge of applications of derivatives while few candidates do not able to differentiate the functions with respect to θ.

Answering Tip

- Give adequate practice on problems based on applications of derivatives.

Detailed Solution:

$$f(\theta) = \frac{4\sin\theta}{2+\cos\theta} - \theta$$

Now, $$f'(\theta) = \frac{(2+\cos\theta)4\cos\theta - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

or $$f'(\theta) = \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

or $$f'(\theta) = \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$ **1**

or $$f'(\theta) = \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

or $$f'(\theta) = \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$\therefore$ $$f'(\theta) = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

or here $f'(\theta)$ is increasing when $f'(\theta) \ge 0$ **1**

i.e., $$\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \ge 0$$

or $$\cos\theta \ge 0$$

$$\left[\because \frac{4-\cos\theta}{(2+\cos\theta)^2} \ge 0 \; \forall \; \theta \in R\right]$$

or $$\theta \in \left[0, \frac{\pi}{2}\right]$$ **1**

$(4 - \cos\theta)$ is always greater than 0.

Since $-1 \le \cos\theta \le 1$, $(2 + \cos\theta)^2 > 0$.

Long Answer Type Questions (5 marks each)

Q. 1. Find the intervals on which the function $f(x) = (x-1)^3(x-2)^2$ is (a) strictly increasing (b) strictly decreasing. R&U [CBSE OD Set I, II, III-2020]

Sol.
$$f(x) = (x-1)^3(x-2)^2$$
$$\Rightarrow \quad f'(x) = (x-1)^2(x-2)(5x-8)$$ **2**

(a) for strictly increasing, $f'(x) > 0$

$\Rightarrow (x-1)^2(x-2)(5x-8) > 0$

$\Rightarrow (x-2)(5x-8) > 0$ (as $x \neq 1$)

$\Rightarrow x \in \left(-\infty, \frac{8}{5}\right) \cup (2, \infty)$ (as $x \neq 1$)

$\therefore \quad x \in (-\infty, 1) \cup \left(1, \frac{8}{5}\right) \cup (2, \infty)$ **1½**

(b) for strictly decreasing, $f'(x) < 0$

$\Rightarrow \quad x \in \left(\frac{8}{5}, 2\right)$ **1½**

[CBSE Marking Scheme 2020] (Modified)

Detailed Solution:

Given, $f(x) = (x-1)^3 (x-2)^2$

On differentiating both sides w.r.t. x, we get

$$f'(x) = (x-1)^3 . \frac{d}{dx}(x-2)^2$$

$$+(x-2)^2 . \frac{d}{dx}(x-1)^3 \quad \left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$f'(x) = (x-1)^3 . 2(x-2) + (x-2)^2 . 3(x-1)^2$$

$$= (x-1)^2 (x-2) [2(x-1) + 3(x-2)]$$

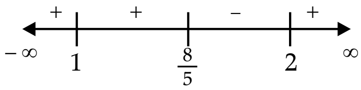
$$= (x-1)^2 (x-2) (2x - 2 + 3x - 6)$$

or, $\quad f'(x) = (x-1)^2 (x-2) (5x-8)$

Now, put $f'(x) = 0$

or, $(x-1)^2 (x-2) (5x-8) = 0$

Either $(x-1)^2 = 0$ or $x - 2 = 0$ or $5x - 8 = 0$



$\therefore \quad x = 1, \frac{8}{5}, 2$

Now, we find intervals and check in which interval $f(x)$ is strictly increasing and strictly decreasing.

Interval	$f'(x) = (x-1)^2 (x-2)(5x-8)$	Sign of $f'(x)$
$x < 1$	(+)(–)(–)	+ve
$1 < x < \frac{8}{5}$	(+)(–)(–)	+ve
$\frac{8}{5} < x < 2$	(+)(–)(+)	–ve
$x > 2$	(+)(+)(+)	+ve

We know that, a function $f(x)$ is said to be an strictly increasing function, if $f'(x) > 0$ and strictly decreasing if $f'(x) < 0$. So, the given function $f(x)$ is increasing on the intervals $(-\infty, 1)$ $\left(1, \frac{8}{5}\right)$ or $(2, \infty)$ and decreasing on $\left(\frac{8}{5}, 2\right)$.

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 1, \frac{8}{5}, 2.$ Hence, $f(x)$ is

(a) increasing on intervals $\left(-\infty, \frac{8}{5}\right) \cup [2, \infty)$

(b) decreasing on interval $\left(\frac{8}{5}, 2\right)$

Commonly Made Error

- Some students expand the polynomial before finding the derivative which is not the actual method.

Answering Tips

- Apply product rule to find the derivatives and for factorization.

Q. 2. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing. R&U **[Delhi Set I, II, III, 2016]**

Topic-3 Maxima and Minima

Concepts Covered • *Local Maxima,* • *Local Minima,* • *Absolute Maxima,* • *Absolute Minima,* • *First derivative test,* • *Second derivative test*

Revision Notes

1. Understanding maxima and minima:

Consider $y = f(x)$ be a well defined function on an **interval** I, then

Key Word

Interval: In mathematics, an interval is a set of real numbers between two given numbers called end points of the interval.

(a) f is said to have a **maximum value** in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$.

The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the **point of maximum value of** f **in** I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$.

The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of** f **in** I.

(c) f is said to have an **extreme value** in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.

Know the Terms

1. Let f be a real valued function and also take a point c from its **domain**, then

> ### Key Word
> **Domain:** The domain refers to the set of possible input values, the domain of a graph consists of all input values shown on the x-axis.

(i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local maximum value of** f.

(ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local minimum value of** f.

2. Critical points

It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes *i.e.*, $f'(c) = 0$ or f is not differentiable.

3. First Derivative Test:

Consider $y = f(x)$ be a well defined function on an open interval I. Now proceed as have been mentioned in the following algorithm:

STEP 1: Find $\frac{dy}{dx}$.

STEP 2: Find the critical point(s) by putting $\frac{dy}{dx} = 0$.

Suppose $c \in I$ (where I is the interval) be any critical point point and f be continuous at this point c. Then we may have following situations :

➲ $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.

➲ $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.

➲ $\frac{dy}{dx}$ **does not change sign** as x increases through c, then $x = c$ is **neither** a point of **local maximum nor** a point of **local minimum**.

Rather in this case, the point $x = c$ is called the **point of inflection**.

4. Second Derivative Test:

Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that:

➲ $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.
The value $f(c)$ is called the local maximum value of f.

➲ $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
The value $f(c)$ is called the local minimum value of f.

This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed above**.**

5. Absolute maxima and absolute minima:

If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

ALGORITHM

STEP 1: Find all the critical points of f in the given interval, *i.e.*, find all the points x where either $f'(x) = 0$ or f is not differentiable.

STEP 2: Take the end points of the given interval.

STEP 3: At all these points (*i.e.*, the points found in STEP 1 and STEP 2) calculate the values of f.

STEP 4: Identify the maximum and minimum value of f out of the values calculated in STEP 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly absolute minimum value is called as **global minimum value** or the **least value.**

> ### Key Facts
> - Rate of change itself is something that we use daily, like comparing one's salary, the weather or even how long it takes for a car to arrive some place.
> - When a cycle moves along a road, then the road becomes the tangent at each point when the wheel rolls on it.
> - Maxima and minina is used to solve optimization problems such as maximizing profit, minimizing the amount of material used in manufacturing or finding the maximum height a rocket can reach.

OBJECTIVE TYPE QUESTIONS

Multiple Choice Questions

Q. 1. A function $f : R \to R$ is defined as $f(x) = x^3 + 1$. Then the function has

(A) no minimum value
(B) no maximum value
(C) both maximum and minimum values
(D) neither maximum value nor minimum value

[CBSE Board 2021]

Ans. Option (D) is correct.

Explanation: Given, $f(x) = x^3 + 1$

$\therefore \quad f'(x) = 3x^2$ and $f''(x) = 6x$

Put $\quad f'(x) = 0$

$\Rightarrow \quad 3x^2 = 0 \Rightarrow x = 0$

At $\quad x = 0, f''(x) = 0$

Thus, $f(x)$ has neither maximum value nor minimum value.

Q. 2. The absolute maximum value of the function $f(x) = 4x - \frac{1}{2}x^2$ in the interval $\left[-2, \frac{9}{2}\right]$ is

(A) 8 **(B)** 9
(C) 6 **(D)** 10 [CBSE Board 2021]

Ans. Option (A) is correct.

Explanation: Given, $f(x) = 4x - \frac{1}{2}x^2$

$\therefore \quad f'(x) = 4 - \frac{1}{2}(2x) = 4 - x$

put $\quad f'(x) = 0$

$\Rightarrow \quad 4 - x = 0$

$\Rightarrow \quad x = 4$

Then, we evaluate the f at critical point $x = 4$ and at the end points of the interval $\left[-2, \frac{9}{2}\right]$.

$$f(4x) = 16 - \frac{1}{2}(16) = 16 - 8 = 8$$

$$f(-2) = -8 - \frac{1}{2}(4)$$

$$= -8 - 2 = -10$$

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2$$

$$= 18 - \frac{81}{8} = 7.875$$

Thus, the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at $x = 4$.

Q. 3. If x is real, the minimum value of $x^2 - 8x + 17$ is

(A) –1 **(B)** 0
(C) 1 **(D)** 2

Ans. Option (C) is correct.

Explanation: Let,

$$f(x) = x^2 - 8x + 17$$

On differentiating with respect to x, we get

$$f'(x) = 2x - 8$$

So, $\quad f'(x) = 0$

$\Rightarrow \quad 2x - 8 = 0$

$\Rightarrow \quad 2x = 8$

$\therefore \quad x = 4$

Now, Again on differentiating with respect to x, we get

$$f''(x) = 2 > 0, \forall x$$

So, $x = 4$ is the point of local minima.

Minimum value of $f(x)$ at $x = 4$

$$f(4) = 4{\cdot}4 - 8{\cdot}4 + 17 = 1$$

Q. 4. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(A) two points of local maximum
(B) two points of local minimum
(C) one maxima and one minima
(D) no maxima or minima

Ans. Option (C) is correct.

Explanation: We have,

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$

$$f'(x) = 6x^2 - 6x - 12$$

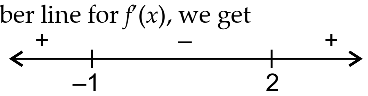
Now, $\quad f'(x) = 0$

$\Rightarrow \quad 6(x^2 - x - 2) = 0$

$\Rightarrow 6(x + 1)(x - 2) = 0$

$\Rightarrow \quad x = -1$ and $x = +2$

On number line for $f'(x)$, we get



Hence, $x = -1$ is point of local maxima and $x = 2$ is point of local minima.

So, $f(x)$ has one maxima and one minima.

Q. 5. The maximum value of $\left(\frac{1}{x}\right)^x$ is :

(A) e **(B)** e^e

(C) $e^{1/e}$ **(D)** $\left(\frac{1}{e}\right)^{1/e}$

[CBSE Board 2021]

Ans. Option (C) is correct.

Explanation:

Let $\quad y = \left(\frac{1}{x}\right)^x$

$\Rightarrow \quad \log y = x.\log\frac{1}{x}$

$\therefore \quad \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\frac{1}{x}}.\left(-\frac{1}{x^2}\right)+\log\frac{1}{x}.1$

$= -1+\log\frac{1}{x}$

$\therefore \quad \frac{dy}{dx} = \left(\log\frac{1}{x}-1\right).\left(\frac{1}{x}\right)^x$

Now, $\quad \frac{dy}{dx} = 0$

$\Rightarrow \quad \log\frac{1}{x} = 1 = \log e$

$\Rightarrow \quad \frac{1}{x} = e$

$\Rightarrow \quad x = \frac{1}{e}$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Find the smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9].

Sol. Given that, the smallest value of polynomial is $f(x)$

$= x^3 - 18x^2 + 96x$

On differentiating with respect to x, we get

$$f'(x) = 3x^2 - 36x + 96$$

So,

$$f'(x) = 0$$

$\Rightarrow \quad 3x^2 - 36x + 96 = 0$

$\Rightarrow 3(x^2 - 12x + 32) = 0$

$\Rightarrow \quad (x-8)(x-4) = 0$

$\Rightarrow \quad x = 8, 4 \in [0, 9]$

We shall now calculate the value of $f(x)$ at these points and at the end points of the interval [0, 9], *i.e.*, at $x = 4$ and $x = 8$ and at $x = 0$ and at $x = 9$.

$$f(4) = 4^3 - 18\times 4^2 + 96\times 4$$
$$= 64 - 288 + 384$$
$$= 160$$
$$f(8) = 8^3 - 18\times 8^2 + 96\times 8$$
$$= 512 - 1152 + 768$$
$$= 128$$
$$f(9) = 9^3 - 18\times 9^2 + 96\times 9$$
$$= 729 - 1458 + 864$$
$$= 135$$
and $f(0) = 0^3 - 18\times 0^2 + 96\times 0$
$$= 0$$

Thus, we conclude that absolute minimum value of $f(x)$ in [0, 9] is 0 occurring at $x = 0$. **1**

Q. 2. Find the maximum value of sin x. cos x.

Q. 3. Find the maximum slope of the curve $y^2 = -x^3 + 3x^2 + 9x - 27$.

Sol. Given that,

$$y = -x^3 + 3x^2 + 9x - 27$$

$\therefore \quad \frac{dy}{dx} = -3x^2 + 6x + 9$

$=$ Slope of the curve

and $\quad \frac{d^2y}{dx^2} = -6x + 6 = -6(x-1)$

$\therefore \quad \frac{d^2y}{dx^2} = 0$

$\Rightarrow \quad -6(x-1) = 0$

$\Rightarrow \quad x = 1 > 0$

Now, $\quad \frac{d^3y}{dx^3} = -6 < 0$

So, the maximum slope of given curve is at $x = 1$.

$\therefore \quad \left(\frac{dy}{dx}\right)_{(x=1)} = -3\times 1^2 + 6\times 1 + 9$

$= 12$ **1**

Short Answer Type Questions-I (2 marks each)

Q. 1. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question?

A I R&U **[Delhi/OD 2018]**

Sol. Let side of base = x and depth of tank = y

$$V = x^2y \Rightarrow y = \frac{V}{x^2},$$

(V = Quantity of water = constant)

Cost of material is least when area of sheet used is minimum.

A (Surface area of tank) = $x^2 + 4xy = x^2 + \frac{4V}{x}$

$$\frac{dA}{dx} = 2x - \frac{4V}{x^2}$$

$$\frac{dA}{dx} = 0 \qquad \text{½}$$

$$\Rightarrow \quad x^3 = 2V,$$

$$\Rightarrow \quad x^3 = 2x^2y,$$

$$y = \frac{x^3}{2x^2} = \frac{x}{2} \quad [\text{as } V = x^2y] \qquad \text{½}$$

$$\frac{d^2A}{dx^2} = 2 + \frac{8V}{x^3} > 0, \qquad \text{½+½}$$

$\therefore$ Area is minimum, thus cost is minimum when

$$y = \frac{x}{2}$$

Value: Any relevant value. **1**

[CBSE Marking Scheme, 2018] (Modified)

Detailed Solution:

Topper's Answer, 2018

Sol.

Let, length = breadth = x

height = y.

volume = $x \times x \times y$

$K = x^2y$.

$y = \frac{K}{x^2}$

According to question,

$S = x^2 + 4xy$

$S = x^2 + 4x \times \frac{K}{x^2} = x^2 + \frac{4K}{x}$

$\frac{ds}{dx} = 2x - \frac{4K}{x^2} = \frac{2x^3 - 4K}{x^2}$

Put $\frac{ds}{dx} = 0$ for critical points

$\frac{2x^3 - 4K}{x^2} = 0$

$2x^3 - 4K = 0$

$x^3 = 2K$

$x = (2K)^{1/3}$

Now, $\frac{d^2s}{dx^2} = 2 - \frac{4K(-2)}{x^3} = 2 + \frac{8K}{x^3}$

$\left(\frac{d^2s}{dx^2}\right)_{(x = (2K)^{1/3})} = 2 + \frac{8K}{2K} = 2 + 4 = 6$

$\frac{d^2s}{dx^2} > 0$ hence it is minimum.

Now, $y = \frac{K}{x^2}$

$y = \frac{K}{(2K)^{2/3}} = \frac{K \cdot K^{-2/3}}{(2)^{2/3}} = \frac{K^{1/3}}{2^{2/3}}$

$y = \frac{1}{2}\left(\frac{K^{1/3}}{2^{-1/3}}\right) \quad y = K^{1/3} \times 2^{-2/3} \quad y = 2^{-1}(2K^{1/3})$

$y = \frac{1}{2}(2K)^{1/3}$

$\boxed{y = \frac{1}{2}x}$ ✓ ½ Hence Proved.

Value :- Helping in nature

Support to middle class people

cooperative & concern towards poor.

Commonly Made Error

- In most of the cases, candidates do not read the question attentively which results incorrect variable in area.

Answering Tip

- Give adequate practice on mensuration related to concept and problems.

Q. 2. Calculate the adjacent sides of a rectangle with a given perimeter as 100 cm and enclosing the maximum area. U

Sol. Let x and y be the adjacent sides of the rectangle,

$\therefore \quad 2x + 2y = 100 \Rightarrow x + y = 50 \quad$ ...(i)

Let A be the area of rectangle.

$\therefore \quad A = xy \Rightarrow y = \frac{A}{x}$

Using (i), we get

$x + \frac{A}{x} = 50 \quad$ **1**

$\Rightarrow \quad A = 50x - x^2$

$\therefore \quad \frac{dA}{dx} = 50 - 2x \quad$ ½

For maximum area $\frac{dA}{dx} = 0 \Rightarrow 50 - 2x = 0$

$\Rightarrow \quad x = 25$

when $\quad x = 25, y = 50 - 25 = 25 \quad$ ½

Hence, adjacent sides are $x = 25$ and $y = 25$

Q. 3. Find the minimum value of $4e^{2x} + 9e^{-2x}$. A I

Sol. Let $\quad f(x) = 4e^{2x} + 9e^{-2x}$

$\therefore \quad f'(x) = 8e^{2x} - 18e^{-2x}$

Put $\quad f'(x) = 0$

$\Rightarrow \quad 8e^{2x} - 18e^{-2x} = 0$

$\Rightarrow \quad e^{2x} = \frac{3}{2} \Rightarrow x = \log\left(\frac{3}{2}\right)^{\frac{1}{2}} \quad$ **1**

Again $\quad f''(x) = 16e^{2x} + 36e^{-2x} > 0$

Now, $f\left(\log\left(\frac{3}{2}\right)^{\frac{1}{2}}\right) = 4e^{2.\log\left(\frac{3}{2}\right)^{\frac{1}{2}}} + 9 \times e^{-2\log\left(\frac{3}{2}\right)^{1/2}}$

$= 4 \times \frac{3}{2} + 9 \times \frac{2}{3}$

$= 6 + 6$

$= 12 \quad$ **1**

Q. 4. It is given that $x = 2$, the function $x^3 - 12x^2 + kx - 8$ attains maximum value, on the interval [0, 3]. Find the value of k. U

Sol. Let $\quad f(x) = x^3 - 12x^2 + kx - 8$

$\therefore \quad f'(x) = 3x^2 - 24x + k \quad$ **1**

It is given that function attains its maximum value of interval [0, 3] at $x = 2$

$\therefore \quad f'(2) = 0$

$\Rightarrow \quad 3(2)^2 - 24(2) + k = 0$

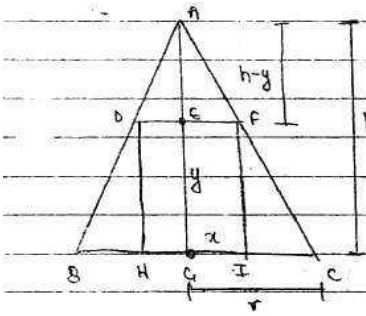
$\Rightarrow \quad k = 36 \quad$ **1**

Long Answer Type Questions (5 marks each)

Q. 1. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone. A I **[CBSE Board 2020]**

Sol.

Topper Answer, 2020



let the cone be represented by ABC and the cylinder have radius and height x and y respectively

~~Volume~~ ∴ As ΔAEF and ΔAGC are similar

$h-y = x$

r

Volume of the cylinder $= V = \pi x^2 y$

$$\Rightarrow V = \frac{\pi x^2}{r}(rh - xh)$$

$$V = \pi x^2 h - \frac{\pi x^3 h}{r}$$

Differentiating with respect to x

$$\Rightarrow \frac{dV}{dx} = 2\pi x h - \frac{3\pi x^2 h}{r} \quad \text{—(i)}$$

for maximum volume $\frac{dV}{dx} = 0$

$$\Rightarrow 2\pi x h = \frac{3\pi x^2 h}{r}$$

$$\Rightarrow 2r = 3x \Rightarrow x = \frac{2}{3}r$$

Differentiating equation (i) with respect to x

$$\Rightarrow \frac{d^2V}{dx^2} = 2\pi h - \frac{6\pi x h}{r}$$

$$\left(\frac{d^2V}{dx^2}\right)_{(x=\frac{2}{3}r)} = 2\pi h - \frac{6\pi 2 r h}{3r} = 2\pi h - 4\pi h = -2\pi h < 0$$

For maximum volume $x = \frac{2r}{3}$

$$y = \frac{rh - xh}{r} = \frac{rh - \frac{2}{3}rh}{r} = \frac{h}{3}$$

∴ Height of the right cylinder with maximum volume is $\frac{1}{3}$rd height of cone.

Volume of the cone $= \frac{1}{3}\pi r^2 h$

Maximum Volume of the cylinder $= \pi x^2 y = \pi \frac{4r^2}{9}\frac{h}{3} = \frac{4\pi r^2 h}{27} = \frac{4}{9}\left(\frac{1}{3}\pi r^2 h\right)$

$$= \frac{4}{9}(\text{Volume of the cone})$$

Hence Proved

Q. 2. Find the minimum value of $(ax + by)$, where $xy = c^2$. A **[CBSE Delhi Set I, III-2020]**

Sol. Let $S = ax + by$, where $y = \frac{c^2}{x}$

$$\therefore \quad S = ax + \frac{bc^2}{x} \qquad 1$$

$$\frac{dS}{dx} = a - \frac{bc^2}{x^2} \qquad 1$$

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad x^2 = \frac{bc^2}{a}$$

$$\text{or} \quad x = \sqrt{\frac{b}{a}}.c \qquad 1$$

$$\left.\frac{d^2S}{dx^2}\right|_{x=\sqrt{\frac{b}{a}}.c} = \frac{2bc^2}{x^3}$$

$$= 2bc^2\left[\sqrt{\frac{a}{b}}\,\frac{1}{c}\right]^3 > 0$$

for $a, b, c > 0$ and $x = \sqrt{\frac{b}{a}}.c$

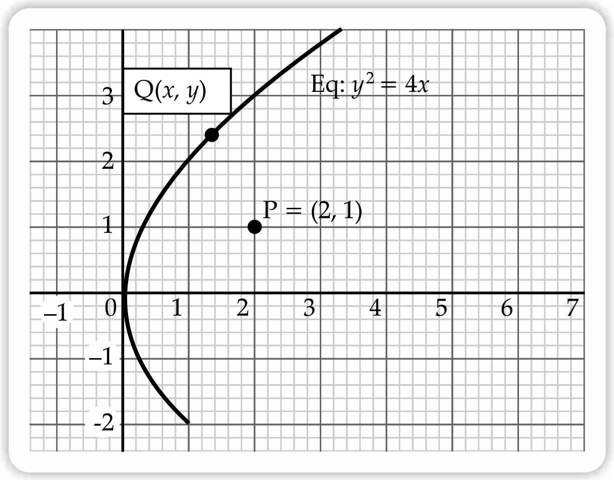
$$\therefore \quad \text{minimum value} = a\sqrt{\frac{b}{a}}.c + b.\frac{c^2}{c}\sqrt{\frac{a}{b}}$$

$$= 2\sqrt{ab}.c$$

[CBSE Marking Scheme 2020] (Modified)

Q. 3. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, 1). A [CBSE Delhi Set-II 2020]

Sol. Let $Q(x, y)$ be the point on curve $y^2 = 4x$, which is nearest to the point P(2, 1)



$$\Rightarrow (PQ)^2 = (x-2)^2 + (y-1)^2 \qquad ½$$

$$\text{Let } (PQ)^2 = S, \text{ then } S = \left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2 \qquad \mathbf{1}$$

$$\frac{dS}{dy} = 2\left(\frac{y^2}{4} - 2\right)\frac{2y}{4} + 2(y-1) = \frac{y^3 - 8}{4} \qquad \mathbf{1}$$

$$\frac{dS}{dy} = 0$$

$$\Rightarrow \quad y = 2$$

$$\therefore \quad x = 1 \qquad \mathbf{1}$$

$$\frac{d^2S}{dy^2} = \frac{3y^2}{4} > 0 \qquad \mathbf{1}$$

$\therefore$ (1, 2) will be at minimum distance from (2, 1). ½

[CBSE Marking Scheme 2020 (Modified)]

Detailed Solution:

Suppose the required point on the curve is $K(p, q)$ and the given point is $A(2, 1)$.

$$\therefore \quad q^2 = 4p \qquad \text{...(i)}$$

$$\text{Then} \quad AK = \sqrt{(p-2)^2 + (q-1)^2}$$

$$\Rightarrow \quad S = \sqrt{\left(\frac{q^2}{4} - 2\right)^2 + (q-1)^2}$$

$$\left[\text{Let } S = AK \text{ from (i) } p = \frac{q^2}{4}\right]$$

$$\text{Then} \quad S^2 = \left(\frac{q^2}{4} - 2\right)^2 + (q-1)^2$$

To find nearest point let us suppose $S^2 = T$

$$\text{Then} \quad T = \left(\frac{q^2}{4} - 2\right)^2 + (q-1)^2$$

$$\Rightarrow \quad T' = 2\left(\frac{q^2}{4} - 2\right) \times \frac{2q}{4} + 2(q-1)$$

For critical points $T' = 0$

$$\Rightarrow \quad 2\left(\frac{q^2}{4} - 2\right) \times \frac{2q}{4} + 2(q-1) = 0$$

$$\Rightarrow \quad \left(\frac{q^3}{4} - 2q\right) + (2q-2) = 0$$

$$\Rightarrow \quad q^3 = 8$$

$$\Rightarrow \quad q = 2 \qquad \text{...(ii)}$$

To find the maxima or minima

$$T'' = \frac{3q^2}{4}$$

$$\Rightarrow \quad T'' \text{ at } (q = 2) = \frac{3 \times 2 \times 2}{4} = 3 > 0$$

Therefore, T is least

From (i) $\quad q^2 = 4p$

$\Rightarrow \quad 2^2 = 4p$

$\Rightarrow \quad p = 1$

Therefore, the required point is $K(1, 2)$.

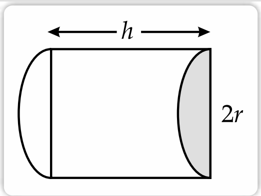
Q. 4. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its side. Also, find the maximum volume.

R&U [CBSE OD Set I, II, III-2020]

Q. 5. A given quantity of metal is to be cast into a solid half circular cylinder with a rectangular base and semi-circular ends. Show that in order that total surface area is minimum, the ratio of length of cylinder to the diameter of semi-circular ends is π : π + 2. AE [CBSE SQP-2020-21]

Sol. Let r be the radius and h be the height of half cylinder

$$\text{Volume} = \frac{1}{2}\pi r^2 h = V(\text{constant}) \qquad \text{...(1)}$$



½

Total surface area of half cylinder is

$$S = 2\left(\frac{1}{2}\pi r^2\right) + \pi rh + 2rh \qquad \text{...(2)}$$

From (1) put the value of h in (2)

$$S = (\pi r^2) + \pi r \left(\frac{2V}{\pi r^2}\right) + 2r\left(\frac{2V}{\pi r^2}\right) \quad \mathbf{1}$$

$$S = (\pi r^2) + \left(\frac{1}{r}\right)\left[\frac{4V}{\pi} + 2V\right]$$

$$\frac{dS}{dr} = (2\pi r) + \left(\frac{-1}{r^2}\right)\left[\frac{4V}{\pi} + 2V\right] \quad \textbf{...(3)}$$

For maxima/minima $\frac{dS}{dr} = 0$

$$\Rightarrow \quad (2\pi r) + \left(\frac{-1}{r^2}\right)\left[\frac{4V}{\pi} + 2V\right] = 0 \quad ½$$

$$\Rightarrow \quad (2\pi r) = \left(\frac{1}{r^2}\right)\left[\frac{4V + 2V\pi}{\pi}\right]$$

$$\Rightarrow \quad \pi r^3 = V\left[\frac{2+\pi}{\pi}\right] \quad ½$$

$$\Rightarrow \quad V = \frac{\pi^2 r^3}{\pi + 2} \quad \textbf{...(4)}$$

From (1) and (4),

$$\Rightarrow \quad \frac{1}{2}\pi r^2 h = \frac{\pi^2 r^3}{\pi+2}$$

$$\Rightarrow \quad \frac{h}{2r} = \frac{\pi}{\pi + 2}$$

$\Rightarrow$ height : diameter = $\pi : (\pi + 2)$ **1**

Differentiating (3) with respect to r

$$\frac{d^2S}{dr^2} = (2\pi) + \left(\frac{2}{r^3}\right)\left[\frac{4V}{\pi} + 2V\right] = \text{positive}$$

(as all quantities are +ve) **1**

So, S is minimum when

height : diameter = $\pi : (\pi + 2)$ Hence Proved ½

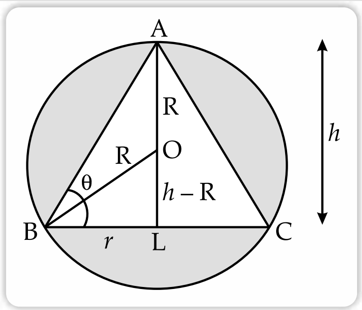
[CBSE SQP Marking Scheme, 2020] (Modified)

Q. 6. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle. AE [CBSE SQP-2020-21]

Sol. Let $2r$ be the base and h be the height of triangle, which is inscribed in a circle of radius R

$$\text{Area of triangle} = \frac{1}{2}\text{(base) (height)}$$

$$\mathbf{A} = \frac{1}{2}(2r)(h) = rh \quad ...(1)$$



½

Area being positive quantity, A will be maximum or minimum if A^2 is maximum or minimum.

$$Z = A^2 = r^2h^2 \quad ...(2)$$

Now in triangle OLB, $BL^2 = OB^2 - OL^2$

In ΔOBD,

$Z = A^2 = r^2h^2$, $r^2 = R^2 - (h - R)^2 \Rightarrow r^2 = 2hR - h^2$

put in (2),

$$Z = h^2(2hR - h^2)$$

$$\Rightarrow \quad Z = (2h^3R - h^4)$$

$$\Rightarrow \quad \frac{dZ}{dh} = 6h^2R - 4h^3 \quad \textbf{...(3)}½$$

For maxima/minima $\frac{dZ}{dh} = 0$

$$\Rightarrow \quad 6h^2R - 4h^3 = 0$$

$$\Rightarrow \quad 6R = 4h \; (h \neq 0)$$

$$\Rightarrow \quad h = \frac{3R}{2} \quad \mathbf{1}$$

Differentiating (3) w.r.t. h

$$\Rightarrow \quad \frac{d^2Z}{dh^2} = 12hR - 12h^2$$

$$\Rightarrow \quad \left.\frac{d^2Z}{dh^2}\right]_{h=\frac{3R}{2}} = 12\left(\frac{3R}{2}\right)R - 12\left(\frac{3R}{2}\right)^2$$

$$= 18R^2 - 27R^2 = -\text{ve}$$

so $Z = A^2$ is maximum when $h = \frac{3R}{2} = 2r^2x - x^3$ **1**

when $h = \frac{3R}{2}$, $r^2 = 2hR - h^2 = 2R.\frac{3R}{2} - \left(\frac{3R}{2}\right)^2$

$$r^2 = \frac{3R^2}{4}$$

$$r = \frac{\sqrt{3}R}{2} \quad \mathbf{1}$$

$$\tan\theta = \frac{h}{r} = \frac{\frac{3R}{2}}{\frac{\sqrt{3}R}{2}} = \sqrt{3}, \theta = \frac{\pi}{3} \quad \mathbf{1}$$

Triangle ABC is equilateral triangle. **Hence Proved.**

[CBSE SQP Marking Scheme, 2020] (Modified)

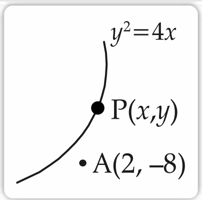
Q. 7. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

A I A [NCERT], [CBSE Delhi Set I-2019], [Delhi Set I, II, III-2016], [Delhi Set I, II, III Comptt. 2016]

Q. 8. Find the point on the curve $y^2 = 4x$, which is nearest to the point (2, – 8).

AE [CBSE OD Set-II, 2019]

Sol. Let $P(x, y)$ be any point on the curve $y^2 = 4x$



$$z = AP = \left|\sqrt{(x-2)^2+(y+8)^2}\right|$$ **1**

let $s = z^2 = \left(\frac{y^2}{4}-2\right)^2+(y+8)^2$

$$\frac{ds}{dy} = 2\left(\frac{y^2}{4}-2\right)\left(\frac{y}{2}\right)+2(y+8) = \frac{y^3}{4}+16$$ **1**

$$\frac{d^2s}{dy^2} = \frac{3y^2}{4}$$ **½**

Let $\frac{ds}{dy} = 0 \Rightarrow y^3 = -64 \Rightarrow y = -4$ **1**

$$\left.\frac{d^2s}{dy^2}\right]_{y=-4} = \frac{3(16)}{4} > 0$$ **½**

$\therefore$ s or z is minimum at $y = -4$; $x = \frac{y^2}{4} = 4$

$\therefore$ The nearest point is $P(4, -4)$ **1**

[CBSE Marking Scheme, 2019] (Modified)

Detailed Solution:

The equation of the given curve is $y^2 = 4x$

$\therefore$ $$x = \frac{y^2}{4}$$ ...(i)

Let $P(x, y)$ be a point on the curve, which is nearest to point $A(2, -8)$.

Now, distance between the points A and P is given by :

$$AP = \left|\sqrt{(x-2)^2+(y+8)^2}\right|$$

$$= \left|\sqrt{\left(\frac{y^2}{4}-2\right)^2+(y+8)^2}\right|$$

... using equ. (i)

$$= \left|\sqrt{\left(\frac{y^4}{16}-y^2+4\right)+(y^2+16y+64)}\right|$$

$$= \left|\sqrt{\frac{y^4}{16}+16y+68}\right|$$

Let $$z = AP^2 = \frac{y^4}{16}+16y+68$$

Now,

$$\frac{dz}{dy} = \frac{1}{16}\times 4y^3+16$$

$$= \frac{y^3}{4}+16$$

For maximum or minimum value of z, we have

$$\frac{dz}{dy} = 0$$

$\Rightarrow$ $$\frac{y^3}{4}+16 = 0$$

$\Rightarrow$ $y^3 + 64 = 0$

$\Rightarrow$ $(y + 4)(y^2 - 4y + 16) = 0$

$\Rightarrow$ $y = -4$

[$\because$ $y^2 - 4y + 16 = 0$ gives imaginary values of y]

Now, $$\frac{d^2z}{dy^2} = \frac{1}{4}\times 3y^2 = \frac{3}{4}y^2$$

for $$\left|\frac{d^2z}{dy^2}\right|_{y=-4} = \frac{3}{4}(-4)^2 = 12 > 0$$

Thus, z is minimum when $y = -4$

Substituting $y = -4$ in the equation of the curve $y^2 = 4x$, we obtain $x = 4$.

Hence, the point $(4, -4)$ on the curve $y^2 = 4x$ is nearest to the point $(2, -8)$.

Q. 9. Prove that the height of the cylinder of maximum volume, that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

A I A [NCERT] [CBSE Delhi Set II-2019]

Sol. Let the radius and height of cylinder be r and h respectively

$\therefore$ $V = \pi r^2 h$ ...(i)

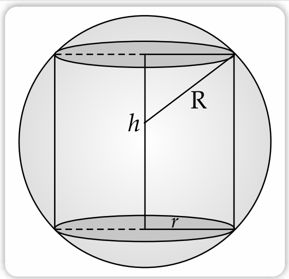
But $r^2 = R^2 - \frac{h^2}{4}$ **½**

$\therefore$ $$\pi h\left(R^2-\frac{h^2}{4}\right) = \pi\left(R^2h-\frac{h^3}{4}\right)$$ **1**

or $$\frac{dV}{dh} = \pi\left(R^2-\frac{3h^2}{4}\right)$$ **½**

For maximum or minimum

$\therefore$ $$\frac{dV}{dh} = 0 \text{ or } h^2 = \frac{4R^2}{3}$$



or $$h = \frac{2R}{\sqrt{3}}$$ **1**

and $$\frac{d^2V}{dh^2} = \pi\left(-\frac{6h}{4}\right) < 0$$ **1**

$$\text{Maximum volume} = \pi\cdot\left[R^2\cdot\frac{2R}{\sqrt{3}} - \frac{1}{4}\left(\frac{2R}{\sqrt{3}}\right)^3\right]$$

$$= \frac{4\pi R^3}{3\sqrt{3}} \text{ cubic units}$$ **1**

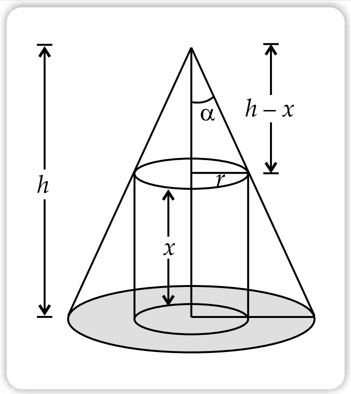
Hence Proved.

[CBSE Marking Scheme, 2019] (Modified)

Q. 10. A cuboidal shaped godown with square base is to be constructed. Three times as much cost per square meter is in curved for constructing the roof as compared to the walls. Find the dimensions of the godown if it is to enclose a given volume and minimize the cost of constructing the roof and the walls. R&U [SQP 2018-19]

Q. 11. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi-vertical angle α, is one-third that of the cone. Hence find the greatest volume of the cylinder. A [NCERT][Delhi Comptt., 2017]

Sol.



1

$$\frac{r}{h-x} = \tan\alpha$$

$$r = (h-x)\tan\alpha$$ ½

Volume of cylinder

$$V = \pi r^2 x$$

$$V = \pi(h-x)^2 x \tan^2\alpha$$ ½

$$\frac{dV}{dx} = \pi\tan^2\alpha\,\{(h-x)^2 + 2x(h-x)(-1)\}$$

$$= \pi\tan^2\alpha\,(h-x)(h-x-2x)$$

$$\frac{dV}{dx} = \pi\tan^2\alpha\,(h-x)(h-3x)$$

$$\frac{dV}{dx} = 0 \text{ or } h = x \text{ or } h = 3x$$

i.e., $$x = \frac{h}{3}$$ **1**

$$\frac{d^2V}{dx^2} = \pi\tan^2\alpha(6x-4h)$$

$\therefore$ $$\frac{d^2V}{dx^2} < 0 \text{ at } x = \frac{h}{3}$$ **1**

$\therefore$ V is maximum, at $x = \frac{h}{3}$

and maximum volume is $V = \frac{4}{27}\pi h^3 \tan^2\alpha$

Hence Proved. 1

[CBSE Marking Scheme, 2017] (Modified)

Q. 12. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum ? A [Foreign 2017]

Sol. Let the length of one piece be x m, then length of the other piece = $(34 - x)$ m

$\therefore$ Side of square is $\frac{x}{4}$ m and perimeter of rectangle

$$2\left(l + \frac{l}{2}\right) = 34 - x$$

Now, $$\text{Area } (A) = \left(\frac{x}{4}\right)^2 + 2\left(\frac{34-x}{6}\right)^2$$ ½

or $$\frac{dA}{dx} = \frac{x}{8} - \frac{1}{9}(34-x)$$ **1**

$$\frac{dA}{dx} = 0 \text{ or } x = 16$$ ½

also, $$\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{1}{9} > 0$$

so, A is minimum when $x = 16$ **1**

$\therefore$ Lengths of the two pieces are 16 m and 18 m. **1**

[CBSE Marking Scheme, 2017] (Modified)

COMPETENCY BASED QUESTIONS

Case based MCQs (4 marks each)

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

In a residential society comprising of 100 houses, there were 60 children between the ages of 10-15 years. They were inspired by their teachers to start composting to ensure that biodegradable waste is recycled, For this purpose, instead of each child doing it for only his/her house, children convinced the Residents welfare association to do it as a society initiative. For this they identified a square area in the local park. Local authorities charged amount of ₹50 per square metre for space so that there is no misuse of the space and Resident welfare association takes it seriously. Association hired a labourer for digging out 250 m^3 and he charged ₹400 x $(\text{depth})^2$. Association will like to have minimum cost.

[CBSE Board 2021]

Q. 1. Let side of square plot is x m and its depth is h metres, then cost C for the pit is

(A) $\frac{50}{h}+400h^2$ (B) $\frac{12500}{h}+400h^2$

(C) $\frac{250}{h}+h^2$ (D) $\frac{250}{h}+400h^2$

Ans. Option (B) is correct.

Explanation: $C = \frac{250\times 50}{h}+400\times h^2$

$\Rightarrow$ $C = \frac{12500}{h}+400h^2$

Q. 2. Value of h (in m) for which $\frac{dC}{dh}=0$ is

(A) 1.5 (B) 2

(C) 2.5 (D) 3

Ans. Option (C) is correct.

Explanation: $C = \frac{12500}{h}+400h^2$

$\therefore$ $\frac{dC}{dh} = \frac{-12500}{h^2}+800h$

Put $\frac{dC}{dh} = 0$

$\therefore$ $\frac{-12500}{h^2}+800h = 0$

$\Rightarrow$ $800h^3 = 12500$

$\Rightarrow$ $h^3 = \frac{125}{8}$

$\Rightarrow$ $h = \frac{5}{2} = 2.5$ m

Q. 3. $\frac{d^2C}{dh^2}$ is given by

(A) $\frac{25000}{h^3}+800$ (B) $\frac{500}{h^3}+800$

(C) $\frac{100}{h^3}+800$ (D) $\frac{500}{h^3}+2$

Ans. Option (A) is correct.

Explanation:

$\therefore$ $\frac{dC}{dh} = \frac{-12500}{h^2}+800h$

$\therefore$ $\frac{d^2C}{dh^2} = \frac{-(-2)\times 12500}{h^3}+800$

$\Rightarrow$ $\frac{d^2C}{dh^2} = \frac{25000}{h^3}+800$

Q. 4. Value of x (in m) for minimum cost is

(A) 5 (B) $10\sqrt{\frac{5}{3}}$

(C) $5\sqrt{5}$ (D) 10

Ans. Option (D) is correct.

Explanation: For minimum cost, put $\frac{dC}{dh} = 0$, we get

$h = 2.5$ m

At $h = 2.5, \frac{d^2C}{dh^2} > 0$

(Hence, minimum)

Value of x at minimum cost

$x = \frac{400\times(2.5)^2}{250}$

$= \frac{2500}{250} = 10$ m

Q. 5. Total minimum cost of digging the pit (in ₹) is

(A) 4,100 (B) 7,500

(C) 7,850 (D) 3,220

Ans. Option (B) is correct.

Explanation: Total minimum cost,

$C = \frac{12500}{h}+400h^2$ (At 2.5)

$\Rightarrow$ $C = \frac{12500}{2.5}+400(2.5)^2$

$\Rightarrow \quad C = 5000 + 2500$

$\Rightarrow \quad C = ₹7500$

II. Read the following text and answer the following questions on the basis of the same:

The shape of a toy is given as $f(x) = 6(2x^4 - x^2)$. To make the toy beautiful 2 sticks which are perpendicular to each other were placed at a point (2, 3), above the toy. **[CBSE QB-2021]**

Q. 1. Which value from the following may be abscissa of critical point?

(A) ± 1/4 **(B)** ± 1/2

(C) ± 1 **(D)** None of these

Ans. Option (B) is correct.

Q. 2. Find the slope of the normal based on the position of the stick.

(A) 360 **(B)** –360

(C) $\frac{1}{360}$ **(D)** $\frac{-1}{360}$

Ans. Option (D) is correct.

Explanation: Slope of the normal based on the position of the stick

$$= \frac{-1}{f'(x)}$$

$$f'(x) = 6\,[8x^3 - 2x]$$
$$f'(2) = 6\,[8 \times 8 - 2 \times 2]$$
$$= 6\,[64 - 4]$$
$$= 360$$
$$\therefore \quad \text{Slope} = \frac{-1}{360}$$

Q. 3. What will be the equation of the tangent at the critical point if it passes through (2, 3)?

(A) $x + 360y = 1082$

(B) $y = 360x - 717$

(C) $x = 717y + 360$

(D) None of these

Ans. Option (B) is correct.

Explanation: We have

$$\left.\frac{dy}{dx}\right]_{(2,\,3)} = 360$$

$$\therefore \quad (y - y') = \frac{dy}{dx}(x - x')$$
$$(y - 3) = 360\,(x - 2)$$
$$y - 3 = 360x - 720$$
$$y = 360x - 717$$

Q. 4. Find the second order derivative of the function at $x = 5$.

(A) 598 **(B)** 1,176

(C) 3,588 **(D)** 3,312

Ans. Option (C) is correct.

Explanation:

$$f(x) = 6(2x^4 - x^2)$$
$$f'(x) = 6[8x^3 - 2x]$$
$$f''(x) = 6[24x^2 - 2]$$
$$f''(5) = 6[24 \times 25 - 2]$$
$$= 6[600 - 2]$$
$$= 3588$$

Q. 5. At which of the following intervals will $f(x)$ be increasing?

(A) $\left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ **(B)** $\left(\frac{-1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

(C) $\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ **(D)** $\left(-\infty, \frac{-1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

Ans. Option (B) is correct.

Explanation: For increasing

$$f'(x) > 0$$
$$6(8x^3 - 2x) > 0$$
i.e., $$x(4x^2 - 1) > 0$$
$$\Rightarrow \quad 4x^2 - 1 > 0$$
and $$x > 0$$
$$4x^2 > 1$$
$$\Rightarrow \quad x^2 > \frac{1}{4}$$
$$\Rightarrow \quad x > \frac{1}{2}$$
and $$x > -\frac{1}{2}$$

i.e., $x \in \left(\frac{-1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

III. Read the following text and answer the following questions, on the basis of the same:

The relation between the height of the plant (y in cm) with respect to exposure to sunlight is governed by the following equation $y = 4x - \frac{1}{2}x^2$ where x is the number of days exposed to sunlight.

[CBSE QB 2021]

Q. 1. The rate of growth of the plant with respect to sunlight is ______ .

(A) $4x-\left(\frac{1}{2}\right)x^2$ **(B)** $4-x$

(C) $x-4$ **(D)** $x-\frac{1}{2}x^2$

Ans. Option (B) is correct.

Explanation:

$$y = 4x-\frac{1}{2}x^2$$

∴ rate of growth of the plant with respect to sunlight

$$= \frac{dy}{dx}$$
$$= \frac{d}{dx}\left[4x-\frac{1}{2}x^2\right]$$
$$= (4-x)\text{ cm / day}$$

Q. 2. What is the number of days it will take for the plant to grow to the maximum height?

(A) 4 **(B)** 6

(C) 7 **(D)** 10

Ans. Option (A) is correct.

Explanation:

$$\frac{dy}{dx} = 4-x$$

The number of days it will take for the plant to grow to the maximum height,

$$\frac{dy}{dx} = 0$$
$$4-x = 0$$
$$x = 4\text{ days.}$$

Q. 3. What is the maximum height of the plant?

(A) 12 cm **(B)** 10 cm

(C) 8 cm **(D)** 6 cm

Ans. Option (C) is correct.

Explanation: We have, number of days for maximum height of plant

$$= 4\text{ Days}$$

∴ Maximum height of plant

$$\Rightarrow \quad y_{(x=4)} = 4\times4-\frac{1}{2}\times4\times4$$
$$= 16-8$$
$$= 8\text{ cm}$$

Q. 4. What will be the height of the plant after 2 days?

(A) 4 cm **(B)** 6 cm

(C) 8 cm **(D)** 10 cm

Ans. Option (B) is correct.

Explanation: Height of plant after 2 days

$$= y_{(x=2)} = 4\times2-\frac{1}{2}\times2\times2$$
$$= 8-2$$
$$= 6\text{ cm}$$

Q. 5. If the height of the plant is 7/2 cm, the number of days it has been exposed to the sunlight is ______ .

(A) 2 **(B)** 3

(C) 4 **(D)** 1

Ans. Option (D) is correct.

Explanation:

Given, $y = \frac{7}{2}$

i.e., $4x-\frac{1}{2}x^2 = \frac{7}{2}$

$$8x-x^2 = 7$$
$$x^2-8x+7 = 0$$
$$x^2-7x-x+7 = 0$$
$$x(x-7)-(x-7) = 0$$
$$x = 1, 7$$

We will take $x = 1$, because it will take 4 days for the plant to grow to the maximum height i.e. 8 cm and $\frac{7}{2}$ cm is not maximum height so, it will take less than 4 days. *i.e.*, 1 day.

IV. Read the following text and answer the following questions, on the basis of the same:

P(x) = $-5x^2 + 125x + 37500$ is the total profit function of a company, where x is the production of the company. **[CBSE QB 2021]**

Q. 1. What will be the production when the profit is maximum?

(A) 37,500 **(B)** 12.5

(C) – 12.5 **(D)** – 37,500

Ans. Option (B) is correct.

Explanation: We, have

$$P(x) = -5x^2 + 125x + 37500$$
$$P'(x) = -10x + 125$$

For maximum profit

$$P'(x) = 0$$
$$-10x + 125 = 0$$
$$-10x = -125$$
$$x = \frac{125}{10}$$
$$= 12.5$$

Q. 2. What will be the maximum profit?

(A) ₹ 38,28,125 **(B)** ₹ 38,281.25
(C) ₹ 39,000 **(D)** None of these

Ans. Option (B) is correct.

Explanation: Maximum profit

$$= P\ (12.5)$$
$$= -5\ (12.5)^2 + 125 \times 12.5 + 37500$$
$$= -781.25 + 1562.5 + 37500$$
$$= 38,281.25$$

Q. 3. Check in which interval the profit is strictly increasing.

(A) (12.5, ∞)
(B) for all real numbers
(C) for all positive real numbers
(D) (0, 12.5)

Ans. Option (D) is correct.

Q. 4. When the production is 2 units what will be the profit of the company?

(A) 37,500 **(B)** 37,730
(C) 37,770 **(D)** None of these

Ans. Option (B) is correct.

Explanation: When production is 2 units, then profit of company = $P(2)$

$$= -5 \times 2^2 + 125 \times 2 + 37500$$
$$= -20 + 250 + 37500$$
$$= 37,730$$

Q. 5. What will be production of the company when the profit is ₹ 38,250?

(A) 15
(B) 30
(C) 10
(D) data is not sufficient to find

Ans. Option (C) is correct.

Explanation:

$$\text{Profit} = 38,250$$
$$\textit{i.e.,}\quad -5x^2 + 125x + 37,500 = 38,250$$
$$5x^2 - 125x + 750 = 0$$
$$x^2 - 25x + 150 = 0$$
$$x\ (x - 15) - 10\ (x - 15) = 0$$
$$(x - 10)\ (x - 15) = 0$$
$$x = 10, 15$$

$$P(x) = -5x^2 + 125x + 37500$$
$$P(10) = -5 \times 10^2 + 125 \times 10 + 37500$$
$$= -500 + 1250 + 37500$$
$$= ₹\ 38,250$$

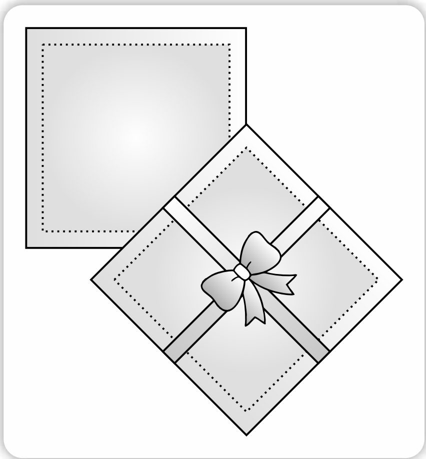
Hence, production of company is 10 units when the profit is ₹38250.

Case based Subjective Questions (4 marks each)

I. Read the following text and answer the following questions on the basis of the same:

(Each Sub-part carries 2 marks)

Revathi is going to her best friend's birthday party. She has made a beautiful painting for her friend as a birthday gift. Now, she wants to prepare a hand made gift box for that painting. For making lower part of the box, she takes a square piece of cardboard of side 30 cm.



Q. 1. If x can be the length of each side of the square cardboard which is to be cut off from corners of the square piece of side 30 cm, then write the expression for volume of the open box formed by folding up the cutting corners.

Also, find $\frac{dV}{dx}$.

Sol. Given, side of square is of length 30 cm.

Also, height of the open box = x cm

Length of open box = $30 - 2x$

and width of open box = $30 - 2x$

∴ Volume (V) of open box = $x \times (30 - 2x) \times (30 - 2x)$

$= x(30 - 2x)^2$

Now, $\frac{dV}{dx} = x[2(30 - 2x)(-2)] + 1(30 - 2x)^2$ **1**

or, $\frac{dV}{dx} = -4x(30 - 2x) + (30 - 2x)^2$

or, $\frac{dV}{dx} = (30 - 2x)\ (-4x + 30 - 2x)$

or, $\frac{dV}{dx} = (30 - 2x)\ (30 - 6x)$ **1**

Q. 2. Revathi is inserted in maximising the volume of the box. So, what should be the side of square to be cut off so that volume of the box is maximum?

Sol. Since, $\frac{dV}{dx} = (30-2x)(30-6x)$ [from above]

$\therefore \quad \frac{d^2V}{dx^2} = -2(30-2x) + (30-2x)(-6)$

or, $\quad \frac{d^2V}{dx^2} = -2(30-6x+90-6x)$

or, $\quad \frac{d^2V}{dx^2} = -2(120-12x)$

or, $\quad \frac{d^2V}{dx^2} = -24(10-x)$

Now, put $\frac{dV}{dx} = 0$

$\Rightarrow (30-2x)(30-6x) = 0$

$\Rightarrow \quad x = 15$ or $x = 5$

At $x = 15$, $\quad \frac{d^2V}{dx} = -24(10-15) = -24 \times (-5)$ **1**

$= 120 > 0$

At $x = 5$, $\quad \frac{d^2V}{dx} = -24(10-15) = -24 \times 5$

$= -120 < 0$ **1**

So, volume will be maximum when $x = 5$.

Solutions for Practice Questions (Topic-1)

Very Short Answer Type Questions

3. Here, $R(x) = 3x^2 + 36x$, where x is the number of sets of books

$$\frac{dR}{dx} = 6x + 36$$

$$\Rightarrow \left(\frac{dR}{dx}\right)_{x=1200} = 6(1200) + 36 = 7,236$$

So the marginal cost of food for 1200 such sets. **1**

$= ₹7,236$

7. $f(x) = x^3 + 6x^2 + 5x + 3$

$$\frac{df(x)}{dx} = 3x^2 + 12x + 5$$

Marginal contentment

$$\left[\frac{df(x)}{dx}\right]_{x=3} = 3 \times (3)^2 + 12 \times 3 + 5$$

$= 27 + 36 + 5$

$= 68$ units **1**

Short Answer Type Questions-I

2. Let x denote the edge of cube, V denote the volume and s denotes the surface area of cube at instant t.

$$\frac{dV}{dt} = 9 \text{ cm}^2/\text{sec}$$

$\because \quad V = x^3$

$\therefore \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$\Rightarrow \quad \frac{9}{3x^2} = \frac{dx}{dt}$ **1**

Now, $\quad s = 6x^2$

$$\frac{ds}{dt} = 12x \times \frac{dx}{dt}$$

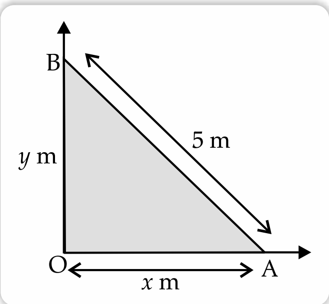
$$\Rightarrow \left(\frac{dx}{dt}\right)_{x=10} = 12 \times 10 \times \frac{9}{3 \times 10 \times 10}$$

$$= \frac{12 \times 9}{3 \times 10} = \frac{36}{10}$$

$= 3.6 \text{ cm}^2/\text{sec}$ **1**

Short Answer Type Questions-II

3. Let AB be the ladder and OB be the wall.



At any instant, let $OA = x$ m and $OB = y$ m

Also, $\quad AB = 5$ m (given)

$\therefore \quad x^2 + y^2 = 5^2 = 25$ ...(i) **1**

Differentiating (i) w.r.t. 't', we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

When $x = 4$, then from (i), we get

$y^2 = 25 - x^2 = 25 - 16 = 9$

$y = 3$

Also, given $\frac{dx}{dt} = 2$ cm/sec $= 0.02$ m/sec **1**

Using these values in (ii), we get

$$4 \times 0.02 + 3\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{0.08}{3} \text{ m/sec} = -\frac{8}{3} \text{ cm/sec}$$

Thus, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ cm/sec. **1**

Long Answer Type Questions

1. Let r be the radius and l be the slant height of water surface

$$\sin\frac{\pi}{4} = \frac{r}{l}$$

or $$r = l.\frac{1}{\sqrt{2}}$$ **1**

Curved surface, $S = \pi rl$

$\therefore$ $$S = \pi.l.\frac{1}{\sqrt{2}}.l$$

$$= \frac{1}{\sqrt{2}}\pi l^2$$ **1**

How, $$\frac{dS}{dt} = \frac{1}{\sqrt{2}}\pi.2l\frac{dl}{dt}$$ **1**

Given that $$\frac{dS}{dt} = -2 \text{ cm}^2/\text{s}$$

$\therefore$ $$-2 = \frac{\pi}{\sqrt{2}}.2l\frac{dl}{dt}$$

$\Rightarrow$ $$\frac{dl}{dt} = \frac{-\sqrt{2}}{\pi l}$$ **1**

At $l = 4$ cm

$$\left(\frac{dl}{dt}\right)_{l=4} = \frac{-\sqrt{2}}{\pi.4} = \frac{-\sqrt{2}}{4\pi} \text{ cm/s}$$ **1**

Commonly Made Error

- Many candidates considered volume as function instead of surface area. Majority of candidates did not apply proper sign through it was specified in the question that find the rate of decrease of slant height of the water.

Answering Tip

- The difference between rate of increase and rate of decrease need to be understood carefully. Practice a number of problems based on application of derivatives.

Solutions for Practice Questions (Topic-2)

Very Short Answer Type Questions

2. Given, $$y = 4x - 9$$

$$\frac{dy}{dx} = 4 > 0 \text{ for all } x \in \text{R}.$$

Hence, function is increasing for all $x \in$ R. **1**

Short Answer Type Questions-I

2. $f'(x) = xe^x$ **1**

Now $x > 0$ and $e^x > 0$ for all x **½**

$\therefore f'(x) > 0 \Rightarrow f$ is an increasing function **½**

[CBSE Marking Scheme 2020]

Detailed Solution:

$$f(x) = (x-1)e^x + 1, x > 0$$
$$f'(x) = (x-1)e^x + e^x(1-0) + 0$$
$$= xe^x - e^x + e^x = xe^x$$

For all $$x > 0$$
$$x.e^x > 0.e^0$$
$$xe^x > 0$$
$$f'(x) > 0$$

Hence, function $f(x) = (x-1)e^x + 1$ is an increasing function for all $x > 0$.

4. $$f(x) = 4x^3 - 18x^2 + 27x - 7$$

$$f'(x) = 12x^2 - 36x + 27$$ **½**

$$= 3(2x-3)^2 \geq 0; x \in R$$ **1**

$\therefore f(x)$ is increasing on R. **½**

[CBSE Marking Scheme, 2017]

Short Answer Type Questions-II

1. $$f(x) = \tan x - 4x$$

$$f'(x) = \sec^2 x - 4$$ **½**

(a) For $f(x)$ to be strictly increasing

$$f'(x) > 0$$

$\Rightarrow$ $$\sec^2 x - 4 > 0$$

$\Rightarrow$ $$\sec^2 x > 4$$

$\Rightarrow$ $$\cos^2 x < \frac{1}{4}$$

$\Rightarrow$ $$\cos^2 x < \left(\frac{1}{2}\right)^2$$

$\Rightarrow$ $$-\frac{1}{2} < \cos x < \frac{1}{2}$$

$\Rightarrow$ $$\frac{\pi}{3} < x < \frac{\pi}{2}$$ **1½**

(b) For $f(x)$ to be strictly decreasing

$$f'(x) < 0$$

$\Rightarrow$ $$\sec^2 x - 4 < 0$$

$\Rightarrow$ $$\sec^2 x > 4$$

$\Rightarrow$ $$\cos^2 x > \frac{1}{4}$$

$\Rightarrow$ $$\cos^2 x > \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \quad \cos x > \frac{1}{2} \qquad \left[\because x \in \left(0, \frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad 0 < x < \frac{\pi}{3} \qquad \mathbf{1}$$

[CBSE SQP Marking Scheme 2020-21]

5. Here, $f'(x) = 3x^2 - 3x^{-4}$

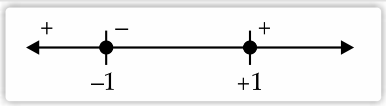
$$= \frac{3(x^6 - 1)}{x^4} \qquad ½$$

$$= \frac{3(x^4 + x^2 + 1)}{x^4}(x+1)(x-1)$$

Critical points are – 1 and 1. ½

or $f'(x) > 0$ if $x > 1$ or $x < -1$, and $f'(x) < 0$ if $-1 < x < 1$

$$\left(\because \frac{3(x^4 + x^2 + 1)}{x^4} \text{always + ve}\right)$$



Hence, $f(x)$ is strictly increasing for $x > 1$ **1**

or $x < -1$; and strictly decreasing in

$(-1, 0) \cup (0, 1)$ **1**

[CBSE SQP Marking Scheme, 2016] (Modified)

Long Answer Type Questions

2. $f(x) = \sin 3x - \cos 3x, \quad 0 < x < \pi$

or $f'(x) = 3\cos 3x + 3\sin 3x$

$= 3(\cos 3x + \sin 3x)$ ½

Put $f'(x) = 0$

or $\cos 3x + \sin 3x = 0$

or $\sin 3x = -\cos 3x$

or $-\tan 3x = 1$

or $\tan 3x = -1$ ½

As $0 < x < \pi$, $0 < 3x < 3\pi$

$\therefore$ $\tan 3x$ is negative for the following values:

$$3x = \frac{3\pi}{4} \text{ or } x = \frac{\pi}{4}$$

$$3x = \pi + \frac{3\pi}{4} = \frac{7\pi}{4}$$

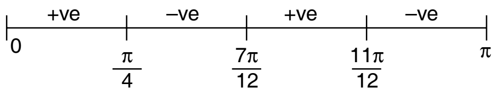
or $x = \frac{7\pi}{12}$

$$3x = \frac{7\pi}{4} + \pi = \frac{11\pi}{4}$$

or $x = \frac{11\pi}{12}$ ½

Hence we have intervals :

Intervals	Sign of $f'(x)$	Nature of function	
$\left(0, \frac{\pi}{4}\right)$	Positive	Increasing	½
$\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$	Negative	Decreasing	½
$\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$	Positive	Increasing	½
$\left(\frac{11\pi}{12}, \pi\right)$	Negative	Decreasing	½



½

Hence, $f(x) = \sin 3x - \cos 3x$ is strictly increasing in the intervals $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ and strictly decreasing in intervals $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$. **1**

Solutions for Practice Questions (Topic-3)

Very Short Answer Type Questions

2. Let us assume that,

$f(x) = \sin x \cos x$

Now, we know that

$$\sin x . \cos x = \frac{1}{2}\sin 2x$$

$$\therefore \quad f'(x) = \frac{1}{2}\cos 2x \times 2 = \cos 2x$$

Now, $f'(x) = 0$

$\Rightarrow \quad \cos 2x = 0$

$\Rightarrow \quad \cos 2x = \cos\frac{\pi}{2}$

$\Rightarrow \quad x = \frac{\pi}{4}$

Also, $f''(x) = \frac{d}{dx} \times \cos 2x = -2 \times \sin 2x$

$$\therefore [f''(x)]_{\text{at } x = \frac{\pi}{4}} = -2\sin 2 \times \frac{\pi}{4}$$

$$= -2\sin\frac{\pi}{2}$$

$$= -2 < 0$$

$\therefore$ $x = \frac{\pi}{4}$ is point of maxima.

$$f\left(\frac{\pi}{4}\right) = \frac{1}{2} \times \sin 2\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

Long Answer Type Questions

4. Let sides of a rectangle are x cm and y cm.

Given $2x + 2y = 36$ **1**

$\Rightarrow \quad y = 18 - x$ **1**

Volume, $V = \pi x^2 y$

$= \pi x^2 (18 - x)$

$= \pi(18x^2 - x^3)$ ½

$$\Rightarrow \frac{dV}{dx} = \pi(36x - 3x^2)$$

For maxima/minima, put $\frac{dV}{dx} = 0$

$\Rightarrow \quad x = 12$ cm ($\because x \neq 0$) ½

Again, $\frac{d^2V}{dx^2} = \pi(36 - 6x)$

$$\Rightarrow \left.\frac{d^2V}{dx^2}\right|_{x=12\text{cm}} = -36\pi < 0$$ **1**

$\therefore$ Volume is maximum when $x = 12$ cm.

Also, $y = (18 - x)$ cm $= 6$ cm

Dimension of rectangle are 12 cm × 6 cm ½

Maximum volume $= \pi x^2 y = 864\pi$ cm^3 ½

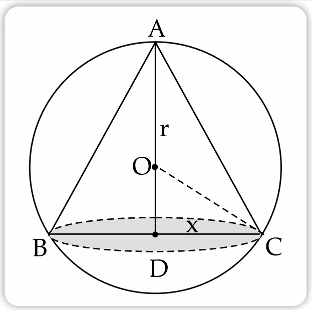
[CBSE Marking Scheme 2020] (Modified)

7. Let radius of cone be x and its height be h.

$\therefore \quad OD = (h - r)$ ½

Volume of cone (V)

$$= \frac{1}{3}\pi x^2 h \quad \text{...(i)}$$ ½



In ΔOCD, $x^2 + (h - r)^2 = r^2$ or $x^2 = r^2 - (h - r)^2$

$$\therefore \quad V = \frac{1}{3}\pi h\{r^2 - (h - r)^2\}$$

$$= \frac{1}{3}\pi(-h^3 + 2h^2 r)$$ **1**

or $$\frac{dV}{dh} = \frac{\pi}{3}(-3h^2 + 4hr)$$ **1**

$$\therefore \quad \frac{dV}{dh} = 0 \text{ or } h = \frac{4r}{3}$$

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h + 4r)$$

$$= \frac{\pi}{3}\left(-6\left(\frac{4r}{3}\right) + 4r\right)$$

$$= -\frac{4\pi r}{3} < 0$$ **1**

$\therefore$ at $h = \frac{4r}{3}$, Volume is maximum.

Maximum volume

$$= \frac{1}{3}\pi\cdot\left\{-\left(\frac{4r}{3}\right)^3 + 2\left(\frac{4r}{3}\right)^2 r\right\}$$

$$= \frac{8}{27}\cdot\left(\frac{4}{3}\pi r^3\right)$$ **1**

$$= \frac{8}{27} \text{ (volume of sphere)}$$

[CBSE Marking Scheme, 2019] (Modified)

10. Let the length and breadth of the base $= x$

Also let the height of the godown $= y$.

Let C be the cost of constructing the godown and V be the given volume. ½

Since cost is proportional to the area, therefore

$$C = k[3x^2 + 4xy],$$

where $k > 0$ is constant of proportionality...(1) **1**

$x^2 y = V$(constant) ...(2) ½

$$y = \frac{V}{x^2} \quad \text{...(3)}$$

Substituting value of y from equation (3), in equation (1), we get

$$C = k\left[3x^2 + 4x\left(\frac{V}{x^2}\right)\right]$$ **1**

$$= k\left[3x^2 + \frac{4V}{x}\right]$$

$$\frac{dC}{dx} = k\left[6x - \frac{4V}{x^2}\right] \quad \text{...(4)}$$ **1**

and $$y = \frac{(18V)^{\frac{1}{3}}}{2}$$ **1**

[CBSE Marking Scheme, 2018] (Modified)

REFLECTIONS

- Do you know where you can use applications of derivatives other than Mathematics?
- Can you easily apply second derivative test to Calculate the maximum/minimum area or volume of geometrical figures?