CHAPTER 5

CONTINUITY & DIFFERENTIABILITY

Syllabus

Continuity and differentiability, chain rule, derivative of inverse trigonometric functions like $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ derivative of implicit functions. Concept of exponential and logarithmic functions. Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives.

In this chapter you will study

- *Continuity of functions*
- *Differentiability of functions like implicit, inverse trigonometric, exponential,logarithmic, etc.*
- *Chain Rule*
- *Higher order derivatives*

List of Topics

Topic-1 Continuity

Concepts Covered • *Left hand Limit,* • *Right Hand Limit*

Revision Notes

FORMULAE FOR LIMITS:

(a) $\lim_{x\to 0} \cos x = 1$

(b) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(c) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

(d) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(e) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$ **(f)** $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, a > 0$

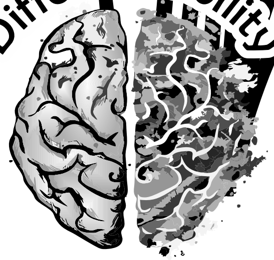
(g) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ **(h)** $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

(i) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

➲ For a function $f(x)$, $\lim_{x\to m} f(x)$ exists if $\lim_{x\to m^-} f(x) = \lim_{x\to m^+} f(x)$.

➲ A function $f(x)$ is continuous at a point $x = m$ if, $\lim_{x\to m^-} f(x) = \lim_{x\to m^+} f(x) = f(m)$, where $\lim_{x\to m^-} f(x)$ is **Left Hand Limit** of $f(x)$ at $x = m$

Continuity and Differentiability



Continuous Function

Suppose f is a real function on a subset of the real numbers and let ' ' be a point in the domain of f.
Then f is continuous at c if $\lim_{x\to c} f(x) = f(c)$

A real function f is said to be continuous if it is continuous at every point in the domain of f.

eg: The function $f(x) = \frac{1}{x}$, $x \neq 0$ is continuous

Let 'c' be any non-zero real number, then $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{1}{x} = \frac{1}{c}$. For $c = 0, f(c) = \frac{1}{c}$ So $\lim_{x\to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of .

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g, f-g, f.g$ and $\frac{f}{g}$ are continuous at $x=c[g(c) \neq 0]$.

Differentiability

Suppose f is a real function and is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

Every differentiable function is continuous, but the converse is not true.

Chain Rule

If $f = vou$, $t = u(x)$ and if both $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt}.\frac{dt}{dx}$.

Logarithmic differentiation

Let $y = f(x) = [u(x)]^{v(x)}$

$\log y = v(x) \log[u(x)]$

$\frac{1}{y}.\frac{dy}{dx} = v(x)\frac{1}{u(x)}u'(x) + v'(x)\log[u(x)]$

$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}u'(x) + v'(x)\log[u(x)]\right]$

e.g. : Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y}.\frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a$.

Derivatives of Implicit functions

If two variables are expressed by some relation then one will be the implicit function of other, is called Implicit function.

For example: Let $y = \cos x - \sin y$, then $\frac{dy}{dx} = \frac{d}{dx}\cos x - \frac{d}{dx}\sin y$

or, $\frac{dy}{dx} = -\sin x - \cos y.\frac{dy}{dx}$ or, $\frac{dy}{dx} = -\sin x / (1+\cos y)$, where $y \neq (2n+1)\pi$

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\left(\frac{dx}{dy} \neq 0\right)$

Thus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ (provided $f'(t) \neq 0$)

eg : if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}f'(x)$ *i.e.*,

$\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y *w.r.t.x.*

eg : if $y = 3x^2+2$, then $y' = 6x$ and $y'' = 6$.

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

(vii) $\frac{d}{dx}(e^x) = e^x$

(viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Trace the Mind Map

- First Level
- Second Level
- Third Level

and $\lim_{x \to m^+} f(x)$ is **Right Hand Limit** of $f(x)$ at $x = m$. Also $f(m)$ is the value of function $f(x)$ at $x = m$.

- A function $f(x)$ is continuous at $x = m$ (say) if, $f(m) = \lim_{x \to m} f(x)$ *i.e.*, a function is continuous at a point in its domain if the **limit value of the function** at that point **equals** the value of the function at the same point.
- For a continuous function $f(x)$ at $x = m$, $\lim_{x \to m} f(x)$ can be directly obtained by evaluating $f(m)$.
- Indeterminate forms or meaningless forms:
  $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 1^{\infty}, 0^0, {}^0.$

OBJECTIVE TYPE QUESTIONS

A Multiple Choice Questions

Q. 1. If a function f defined by

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3 & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

is continuous at $x = \frac{\pi}{2}$, then the value of k is:

(A) 2 **(B)** 3
(C) 6 **(D)** –6

[CBSE Term-I 2021]

Ans. Option (C) is correct.

Explanation: Since, $f(x)$ is continuous at $x = \frac{\pi}{2}$

Therefore, $\lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = 3$$

$$\Rightarrow k \lim_{x \to \frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} = 3$$

$$\Rightarrow \frac{k}{2} \lim_{x \to \frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)} = 3$$

$$\Rightarrow \frac{k}{2} \times 1 = 3 \Rightarrow k = 6$$

Q. 2. The function $f(x) = \begin{cases} \dfrac{e^{3x} - e^{-5x}}{x}, & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$ for the value of k, as:

(A) 3 **(B)** 5
(C) 6 **(D)** 8

[CBSE Term-I 2021]

Ans. Option (D) is correct.

Explanation: Since, $f(x)$ is continuous at $x = 0$, then

LHL = RHL = $f(0)$

or LHL = RHL = k

Now, $$\text{LHL} = \lim_{h \to 0} \frac{e^{3(0-h)} - e^{-5(0-h)}}{0 - h}$$

$$= \lim_{h \to 0} \frac{e^{-3h} - e^{5h}}{-h}$$

$$= \lim_{h \to 0}\left(\frac{e^{-3h} - 1}{-h}\right) + \lim_{h \to 0}\left(\frac{e^{5h} - 1}{h}\right)$$

$$= 3\lim_{h \to 0}\left(\frac{e^{-3h} - 1}{-3h}\right) + 5\lim_{h \to 0}\left(\frac{e^{5h} - 1}{5h}\right)$$

$$= 3 \times 1 + 5 \times 1 = 8$$

Thus, $k = 8$

Q. 3. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ then which of the following can be a discontinuous function?

(A) $f(x) + g(x)$ **(B)** $f(x) - g(x)$
(C) $f(x).g(x)$ **(D)** $\frac{g(x)}{f(x)}$

Ans. Option (D) is correct.

Explanation: Since $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ are continuous functions, then by using the algebra of continuous functions, the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).g(x)$ are also continuous functions but $\frac{g(x)}{f(x)}$ is discontinuous function at $x = 0$.

Q. 4. The function $f(x) = \frac{4 - x^2}{4x - x^3}$ is:

(A) discontinuous at only one point
(B) discontinuous at exactly two points
(C) discontinuous at exactly three points
(D) none of these

Ans. Option (C) is correct.

Explanation: Given that,

$$f(x) = \frac{4 - x^2}{4x - x^3},$$

then it is discontinuous if

$\Rightarrow \quad 4x - x^3 = 0$

$\Rightarrow \quad x(4 - x^2) = 0$

$\Rightarrow x(2+x)(2-x) = 0$

$\Rightarrow \quad x = 0, -2, 2$

Thus, the given function is discontinuous at exactly three points.

Q. 5. The function $f(x) = \cot x$ is discontinuous on the set:

(A) $\{x = n\pi;\ n \in Z\}$ **(B)** $\{x = 2n\pi;\ n \in Z\}$

(C) $\left\{x = (2n+1)\frac{\pi}{2};\ n \in Z\right\}$

(D) $\left\{x = \frac{n\pi}{2};\ n \in Z\right\}$

Ans. Option (A) is correct.

Explanation: Given that,

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

It is discontinuous at

$$\sin x = 0$$

$\Rightarrow \quad x = n\pi, n \in Z$

Thus, the given function is discontinuous at $\{x = n\pi : n \in Z\}$.

Q. 6. If $f(x) = \begin{cases} mx+1 & \text{if } x \le \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$, is continuous at $x = \frac{\pi}{2}$ then

(A) $m = 1, n = 0$ **(B)** $m = \frac{n\pi}{2} + 1$

(C) $n = \frac{m\pi}{2}$ **(D)** $m = n = \frac{\pi}{2}$

Ans. Option (C) is correct.

Explanation: Given that,

$$f(x) = \begin{cases} mx+1 & \text{if } x \le \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$$

is continuous function at $x = \frac{\pi}{2}$, then

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x)$$

$$\Rightarrow \lim_{h\to 0} f\left(\frac{\pi}{2} - h\right) = \lim_{h\to 0} f\left(\frac{\pi}{2} + h\right)$$

$$\Rightarrow \lim_{h\to 0} m\left(\frac{\pi}{2} - h\right) + 1 = \lim_{h\to 0} \sin\left(\frac{\pi}{2} + h\right) + n$$

$$\Rightarrow \lim_{h\to 0} m\left(\frac{\pi}{2} - h\right) + 1 = \lim_{h\to 0} \cos h + n$$

$$\Rightarrow m\left(\frac{\pi}{2}\right) + 1 = 1 + n$$

$$\Rightarrow n = \frac{m\pi}{2}$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. If the function f defined as

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \ne 3 \\ k, & x = 3 \end{cases}$$

is continuous at $x = 3$, find the value of k.

R&U [CBSE Delhi Set-I, II, III 2020]

Q. 2. Determine the value of 'k' for which the following function is continuous at $x = 3$:

$$f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x - 3}, & x \ne 3 \\ k, & x = 3 \end{cases}$$

R&U [OD, 2017]

Q. 3. Determine the value of the constant 'k' so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{cases}$ is continuous at $x = 0$.

R&U [Delhi, 2017]

Sol. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{k(-x)}{|x|} = -k$ ½

$k = -3$

[CBSE Marking Scheme, 2017] ½

Detailed Solution:

Since f is continuous at $x = 0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \quad ½$$

Here $f(0) = 3$,

$$\text{LHL} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-}\frac{kx}{|x|} = \lim_{x\to 0^-}\frac{k(-x)}{|x|} = -k$$

$\therefore$ $-k = 3$ or $k = -3.$ ½

Q. 4. If the following function $f(x)$ is continuous at $x = 0$, then write the value of k.

$$f(x) = \begin{cases} \dfrac{\sin\frac{3x}{2}}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

R&U [OD Comptt., 2017]

Sol. $$\lim_{x\to 0}\frac{\sin\frac{3x}{2}}{x} = \lim_{x\to 0}\frac{3}{2}\cdot\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}$$ ½

or $k = \dfrac{3}{2}$ ½

[CBSE Marking Scheme, 2017]

Short Answer Type Questions-I (2 marks each)

Q. 1. Find the value(s) of k so that the following function is continuous at $x = 0$

$$f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x} & \text{if } x \neq 0 \\ \dfrac{1}{2} & \text{if } x = 0 \end{cases}$$

A I R&U [CBSE SQP 2020-21]

Sol. $$\underset{x\to 0}{Lt}\frac{1-\cos kx}{x\sin x} = \underset{x\to 0}{Lt}\frac{2\sin^2\left(\frac{kx}{2}\right)}{x\sin x}$$

$$= \underset{x\to 0}{Lt}\frac{\dfrac{2\sin^2\left(\frac{kx}{2}\right)}{x^2}}{\dfrac{x\sin x}{x^2}}$$

$$= \frac{\underset{x\to 0}{Lt}\dfrac{2\sin^2\left(\frac{kx}{2}\right)}{\left(\frac{kx}{2}\right)^2}\times\left(\frac{k}{2}\right)^2}{\underset{x\to 0}{Lt}\dfrac{\sin x}{x}}$$

$$= \frac{2\times 1\times\frac{k^2}{4}}{1}$$ 1½

$\because f(x)$ is continuous at $x = 0$

$\therefore$ $\underset{x\to 0}{Lt} f(x) = f(0)$

$\Rightarrow$ $\dfrac{k^2}{2} = \dfrac{1}{2}$

$\Rightarrow$ $k^2 = 1$

$\Rightarrow$ $k = \pm 1$ ½

[CBSE Marking Scheme 2020-21]

Commonly Made Error

- Some students do not know how to evaluate limits of the form $\dfrac{0}{0}$.

Answering Tip

- Learn to evaluate the indeterminate forms of limits.

Q. 2. Find the value of k for which the function.

$$f(x) = \begin{cases} \dfrac{x^2+3x-10}{x-2} & , x \neq 2 \\ k & , x = 2 \end{cases}$$

is continuous at $x = 2$.

R&U [Delhi Comptt. 2017]

Q. 3. Find the value of p for which the function

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & , x \neq 0 \\ p & , x = 0 \end{cases}$$

is continuous at $x = 0$. R&U [Delhi Comptt. 2017]

Sol. $$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0}\frac{4\times 2\sin^2 2x}{4x^2} = p$$ 1

$$8\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = p$$ ½

$$p = 8$$ ½

[CBSE Marking Scheme, 2017]

Q. 4. Find the value of k for which the function

$$f(x) = \begin{cases} \dfrac{\sin x - \cos x}{4x-\pi} & , x \neq \frac{\pi}{4} \\ k & , x = \frac{\pi}{4} \end{cases}$$ **is continuous at $x = \dfrac{x}{4}$.**

R&U [Delhi Comptt. 2017]

Sol. $$\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{4x-\pi} = f\left(\frac{\pi}{4}\right)$$

$$\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right)}{4x-\pi} = k$$

$$\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\left(\sin x\cos\frac{\pi}{4} - \cos x.\sin\frac{\pi}{4}\right)}{4x-\pi} = k$$

$$\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)}{4\left(x-\frac{\pi}{4}\right)} = k$$

$$\frac{\sqrt{2}}{4} = k$$

$$k = \frac{1}{2\sqrt{2}}$$

[CBSE Marking Scheme, 2017]

Short Answer Type Questions-II (3 marks each)

Q. 1. Find the values of p and q, for which

$$f(x)=\begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\dfrac{\pi}{2} \\ p, & \text{if } x=\dfrac{\pi}{2}, \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\dfrac{\pi}{2} \end{cases}$$

is continuous at $x = \frac{\pi}{2}$.

A **[Delhi Set I, II, III 2016]**

Sol. $\underset{\left(\text{at } x=\frac{\pi}{2}\right)}{\text{LHL}} = \underset{x\to\frac{\pi}{2}^-}{\text{Lt}} f(x)$

$$= \underset{x\to\frac{\pi}{2}^-}{\text{Lt}} \frac{(1-\sin^3 x)}{3\cos^2 x}$$

$$= \underset{x\to\frac{\pi}{2}^-}{\text{Lt}} \frac{(1^3-\sin^3 x)}{3(1^2-\sin^2 x)} \qquad ½$$

$$= \underset{x\to\frac{\pi}{2}^-}{\text{Lt}} \frac{(1-\sin x)(1+\sin x+\sin^2 x)}{3(1-\sin x)(1+\sin x)}$$

$(1 - \sin x) \neq 0$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1+\sin x+\sin^2 x}{3(1+\sin x)} \qquad ½$$

Put $x = \frac{\pi}{2} - h$. As $x \to \frac{\pi}{2}^-$, $h \to 0$

$$\lim_{h\to 0} \frac{1+\sin\left(\frac{\pi}{2}-h\right)+\sin^2\left(\frac{\pi}{2}-h\right)}{3\left(1+\sin\left(\frac{\pi}{2}-h\right)\right)}$$

$$\lim_{h\to 0} \frac{1+\cos h+\cos^2 h}{3(1+\cos h)}$$

$$= \frac{1+1+1}{3(1+1)} = \frac{1}{2} \qquad ½$$

$$\underset{\left(\text{at } x=\frac{\pi}{2}\right)}{\text{RHL}} = \underset{\left(x\to\frac{\pi}{2}^+\right)}{\text{Lt}} .f(x)$$

$$= \underset{x\to\frac{\pi}{2}^+}{\text{Lt}} \frac{q(1-\sin x)}{(\pi-2x)^2}$$

Put $x = \frac{\pi}{2}+h$. As $x \to \frac{\pi}{2}, h \to 0$ ½

$$= q \underset{h\to 0}{\text{Lt}} \frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^2}$$

$$= q \underset{h\to 0}{\text{Lt}} \frac{1-\cos h}{4h^2} = q \underset{h\to 0}{\text{Lt}} \frac{2\sin^2 \frac{h}{2}}{4h^2}$$

$$= \frac{q}{2} \underset{h\to 0}{\text{Lt}} \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 \times \frac{1}{4} = \frac{q}{8} \qquad ½$$

Also, $f\left(\frac{\pi}{2}\right) = p$

Since $f(x)$ is continuous at $x = \frac{\pi}{2}$

$\therefore$ LHL $=$ RHL $= f\left(\frac{\pi}{2}\right)$

$\therefore$ $\frac{1}{2} = \frac{q}{8} = p$

or $q = 4$ and $p = \frac{1}{2}$ ½

Commonly Made Error

- Many student commit errors in finding the Left hand limit and Right hand limit.

Answering Tip

- Sufficient time needs to be spent on this topic.

Q. 2. For what value of k is the following function continuous at $x = -\frac{\pi}{6}$?

$$f(x) = \begin{cases} \dfrac{\sqrt{3}\sin x+\cos x}{x+\frac{\pi}{6}}, & x \neq -\dfrac{\pi}{6} \\ k, & x = -\dfrac{\pi}{6} \end{cases}$$

A **[SQP 2016-17]**

Sol.

$$\lim_{x\to -\frac{\pi}{6}} f(x) = \lim_{x\to -\frac{\pi}{6}} \frac{\sqrt{3}\sin x + \cos x}{x+\frac{\pi}{6}}$$

$$\lim_{x\to -\frac{\pi}{6}} f(x) = \lim_{x\to -\frac{\pi}{6}} \frac{2\left(\frac{\sqrt{3}}{2}\sin x + \cos x\cdot\frac{1}{2}\right)}{x+\frac{\pi}{6}}$$

$$\lim_{x\to -\frac{\pi}{6}} f(x) = \lim_{x\to -\frac{\pi}{6}} \frac{2\sin\left(x+\frac{\pi}{6}\right)}{x+\frac{\pi}{6}} \quad \mathbf{1}$$

$$x \to -\frac{\pi}{6} \to -\frac{\pi}{6}+h = 2\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$= 2 \quad \mathbf{1}$$

$$f\left(-\frac{\pi}{6}\right) = k \quad ½$$

For the continuity of $f(x)$ at $x = -\frac{\pi}{6}$,

$$f\left(-\frac{\pi}{6}\right) = \lim_{x\to -\frac{\pi}{6}} f(x)$$

or $k = 2$ ½

[CBSE Marking Scheme, 2016] (Modified)

Q. 3. Determine the values of 'a' such that the following function is continuous at $x = 0$:

$$f(x) = \begin{cases} \dfrac{x+\sin x}{\sin(a+1)x}, & if\ -\pi < x < 0 \\ 2, & if\ x = 0 \\ 2\dfrac{e^{\sin bx}-1}{bx}, & if\ x > 0 \end{cases}$$

A I R&U [SQP, 2017-18]

Sol.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x+\sin x}{\sin(a+1)x}$$

$$= \lim_{x\to 0^-} \frac{1+\frac{\sin x}{x}}{\frac{\sin(a+1)x}{(a+1)x}(a+1)}$$

$$= \frac{2}{a+1} \quad ½$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2\frac{e^{\sin bx}-1}{bx} \quad ½$$

$$= \lim_{x\to 0^+} 2\frac{e^{\sin bx}-1}{\sin bx}\times\frac{\sin bx}{bx} \quad ½$$

$$f(0) = 2. \quad ½$$

For the function to be continuous at 0, we must have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

i.e., we must have $\frac{2}{a+1} = 2$ or $a = 0$; b may be any real number other than 0. **1**

[CBSE SQP Marking Scheme, 2017-18] (Modified)

Q. 4. Find k, if $f(x) = \begin{cases} k\sin\frac{\pi}{2}(x+1), & x \le 0 \\ \dfrac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$ is continuous at $x = 0$. R&U [OD 2016]

Topic-2 Differentiability

Concepts Covered • *Left Hand Derivative,* • *Right Hand Derivative,* • *Relation between Continuity and Differentiability*

Derivative of Some Standard Functions:

(a) $\frac{d}{dx}(x^n) = nx^{n-1}$

(b) $\frac{d}{dx}(k) = 0$, where k is any constant

(c) $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 0$

(d) $\frac{d}{dx}(e^x) = e^x$

(e) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a} = \frac{1}{x}\log_a e$

(f) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(g) $\frac{d}{dx}(\sin x) = \cos x$

(h) $\frac{d}{dx}(\cos x) = -\sin x$

(i) $\frac{d}{dx}(\tan x) = \sec^2 x$

(j) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(k) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(l) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x\, \cot x$

(m) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}, x\in(-1,1)$

(n) $\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}, x\in(-1,1)$

(o) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}, x\in R$

(p) $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}, x\in R$

(q) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$, where $x\in(-\infty,-1)\cup(1,\infty)$

(r) $\frac{d}{dx}(\text{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$, where $x\in(-\infty,-1)\cup(1,\infty)$

Following derivatives should also be memorized by you for quick use:

(i) $\frac{d}{dx}\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$

➲ **Left Hand Derivative of $f(x)$ at $x = m$,**

$$Lf'(m) = \lim_{x\to m^-}\frac{f(x)-f(m)}{x-m} \text{ and,}$$

Right Hand Derivative of $f(x)$ at $x = m$,

$$Rf'(m) = \lim_{x\to m^+}\frac{f(x)-f(m)}{x-m}$$

For a **function to be differentiable** at a point, **LHD and RHD at that point should be equal**.

➲ **Derivative of y *w.r.t.* x:** $\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$.

Also, for very-very small value h, $f'(x) = \frac{f(x+h)-f(x)}{h}$, (as $h\to 0$)

Relation between Continuity and Differentiability:

(i) If a function is differentiable at a point, it is continuous at that point as well.

(ii) If a function is not differentiable at a point, it may or may not be continuous at that point.

(iii) If a function is continuous at a point, it may or may not differentiable at that point.

(iv) If a function is **discontinuous** at a point, it is not be differentiable at that point.

Rules of Derivatives:

➲ Product or Leibniz's rule of derivatives:

$$\frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$$

➲ Quotient Rule of derivatives:

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}=\frac{vu'-uv'}{v^2}.$$

Key Word

Discontinuous Function: A discontinuous function is a function in algebra that has a point where either the function is not defined at that point or the LHL and RHL of the function are equal but not equal to the value of the function at that point or the limit of the function does not exist at the given point.

Key Facts

- All differentiable functions happen to be continuous but not all continuous functions can said to be differentiable.
- A function is said to be continuously differentiable if the derivative exists and is itself a continuous function.
- $f(x) = 0$ is a continuous function because it is an unbroken line, without holes or jumps.
- If $f(0) = \infty$, then function is continuous at 0.
- All polynomial functions are continuous functions.

Mnemonics

Quotient Rule of Derivative

Ho D Hi Minus Hi D Ho Over Ho Ho

In mathematical notation, $\frac{\text{Ho D Hi - Hi D Ho}}{\text{ho ho}}$

where, Ho → function in numerator

Hi → function in denominator

D → derivative of

OBJECTIVE TYPE QUESTIONS

A Multiple Choice Questions

Q. 1. Differential of log[log(log x^5)] w.r.t x is:

(A) $\dfrac{5}{x\log(x^5)\log(\log x^5)}$ **(B)** $\dfrac{5}{x\log(\log x^5)}$

(C) $\dfrac{5x^4}{\log(x^5)\log(\log x^5)}$ **(D)** $\dfrac{5x^4}{\log x^5\log(\log x^5)}$

[CBSE Term-I 2021]

Ans. Option (A) is correct.

Explanation: Let $y = \log[\log(\log x^5)]$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log(\log x^5)}\frac{dy}{dx}[\log(\log x^5)]$$

(By Chain Rule)

$$= \frac{1}{\log(\log x^5)}\cdot\frac{1}{\log x^5}\frac{d}{dx}\log x^5$$

$$= \frac{1}{\log(x^5)\log(\log x^5)}\frac{1}{x^5}\frac{d}{dx}(x^5)$$

$$= \frac{5}{x\log(x^5)\log(\log x^5)}$$

Q. 2. If $(x^2 + y^2)^2 = xy$, then $\dfrac{dy}{dx}$ is:

(A) $\dfrac{y+4x(x^2+y^2)}{4y(x^2+y^2)-x}$ **(B)** $\dfrac{y-4x(x^2+y^2)}{x+4(x^2+y^2)}$

(C) $\dfrac{y-4x(x^2+y^2)}{4y(x^2+y^2)-x}$ **(D)** $\dfrac{4y(x^2+y^2)-x}{y-4x(x^2+y^2)}$

[CBSE Term-I 2021]

Ans. Option (C) is correct.

Explanation: Given, $(x^2 + y^2)^2 = xy$

$\Rightarrow \quad x^4 + 2x^2y^2 + y^4 - xy = 0$

Differentiating w.r.t. x, we get

$$4x^3 + 2\left[2xy^2 + 2x^2y\frac{dy}{dx}\right] + 4y^3\frac{dy}{dx} - \left[y + x\frac{dy}{dx}\right] = 0$$

$$\frac{dy}{dx}\left[4x^2y + 4y^3 - x\right] + \left[4x^3 + 4xy^2 - y\right] = 0$$

$$\frac{dy}{dx} = \frac{-\left[4x^3 + 4xy^2 - y\right]}{\left[4x^2y + 4y^3 - x\right]}$$

$$\text{or } \frac{dy}{dx} = \frac{y - 4x(x^2+y^2)}{4y(x^2+y^2)-x}$$

Q. 3. If $\sin y = x\cos(a + y)$, then $\dfrac{dx}{dy}$ is:

(A) $\dfrac{\cos a}{\cos^2(a+y)}$ **(B)** $\dfrac{-\cos a}{\cos^2(a+y)}$

(C) $\dfrac{\cos a}{\sin^2 y}$ **(D)** $\dfrac{-\cos a}{\sin^2 y}$

[CBSE Term-I 2021]

Ans. Option (A) is correct.

Explanation: Given, $\sin y = x\cos(a + y)$

$$\Rightarrow \quad x = \frac{\sin y}{\cos(a+y)}$$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}(\sin y) - \sin y\frac{d}{dy}\{\cos(a+y)\}}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos(a+y)\cos y - \sin y[-\sin(a+y)]}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos(a+y)\cos y + \sin y\sin(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos[(a+y)-y]}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos a}{\cos^2(a+y)}$$

Commonly Made Error

- Sometimes students commit errors in $\dfrac{u}{v}$ rule for differentiating the function.

Answering Tip

- Differentiation rules for different functions and forms need continuous revision practice.

Q. 4. If $y = \sin(m\sin^{-1}x)$, then which one of the following equations is true ?

(A) $(1-x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + m^2y = 0$

(B) $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2y = 0$

(C) $(1+x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - m^2y = 0$

(D) $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - m^2y = 0$

[CBSE Term-I 2021]

Ans. Option (B) is correct.

Explanation: Given, $y = \sin(m(\sin^{-1}x))$ ...(i)

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \cos(m\sin^{-1}x)\times\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{m\cos(m\sin^{-1}x)}{\sqrt{1-x^2}} \quad \text{...(ii)}$$

$$\Rightarrow \quad y' = \frac{m\cos(m\sin^{-1}x)}{\sqrt{1-x^2}} \quad \text{...(ii)}$$

$$\Rightarrow \left(\sqrt{1-x^2}\right)y' = m\cos(\text{m}\sin^{-1}x)$$

Differentiating again w.r.t. 'x', we get

$$y''\left(\sqrt{1-x^2}\right)+y'\frac{(-2x)}{2\sqrt{1-x^2}}$$

$$= -m^2\sin(m\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow y''(1-x^2) - xy' = -m^2y$

$\Rightarrow y''(1-x^2) - xy' + m^2y = 0$

or $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

Q. 5. If $x = 2\cos\theta - \cos2\theta$ and $y = 2\sin\theta - \sin2\theta$, then $\frac{dy}{dx}$ is:

(A) $\frac{\cos\theta+\cos2\theta}{\sin\theta-\sin2\theta}$ (B) $\frac{\cos\theta-\cos2\theta}{\sin2\theta-\sin\theta}$

(C) $\frac{\cos\theta-\cos2\theta}{\sin\theta-\sin2\theta}$ (D) $\frac{\cos2\theta-\cos\theta}{\sin2\theta+\sin\theta}$

[CBSE Term-I 2021]

Ans. Option (B) is correct.

Explanation: Given, $x = 2\cos\theta - \cos2\theta$

and $y = 2\sin\theta - \sin2\theta$

Therefore, $\frac{dx}{d\theta} = -2\sin\theta + 2\sin2\theta$

and $\frac{dy}{d\theta} = 2\cos\theta - 2\cos2\theta$

$\therefore \quad \frac{dy}{dx} = \frac{2\cos\theta-2\cos2\theta}{-2\sin\theta+2\sin2\theta}$

or $\quad \frac{dy}{dx} = \frac{\cos\theta-\cos2\theta}{\sin2\theta-\sin\theta}$

Q. 6. If $y = Ae^{5x} + Be^{-5x}$, then $\frac{d^2y}{dx^2}$ is equal to:

(A) $25y$ (B) $5y$

(C) $-25y$ (D) $15y$

[CBSE Delhi Set-I 2020]

Ans. Option (A) is correct.

Explanation:

$y = Ae^{5x} + Be^{-5x}$

$\Rightarrow \frac{dy}{dx} = 5Ae^{5x} - 5Be^{-5x}$

$\Rightarrow \frac{d^2y}{d^2x} = 25Ae^{5x} + 25Be^{-5x}$

$= 25y$

Q. 7. If $y = \log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ equals:

(A) $-\frac{1}{x}$ (B) $-\frac{1}{x^2}$

(C) $\frac{2}{x^2}$ (D) $-\frac{2}{x^2}$

[CBSE Delhi Set-III 2020]

Ans. Option (D) is correct.

Explanation:

Given, $y = \log_e\left(\frac{x^2}{e^2}\right)$

$\Rightarrow y = 2\log_e x - \log_e e^2$

$\Rightarrow y = 2\log_e x - 2$

$\Rightarrow \frac{dy}{dx} = \frac{2}{x}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2}{x^2}$

Q. 8. The set of points where the function f given by $f(x) = |2x-1|\sin x$ is differentiable is:

(A) R (B) $R-\left\{\frac{1}{2}\right\}$

(C) $(0, \infty)$ (D) none of these

Ans. Option (B) is correct.

Explanation: Given that,

$f(x) = |2x-1|\sin x$

The function $\sin x$ is differentiable.

The function $|2x-1|$ is differentiable, except

$2x - 1 = 0$

$\Rightarrow x = \frac{1}{2}$

Thus, the given function is differentiable $R-\left\{\frac{1}{2}\right\}$.

Q. 9. The function $f(x) = e^{|x|}$ is

(A) continuous everywhere but not differentiable at $x = 0$

(B) continuous and differentiable everywhere

(C) not continuous at $x = 0$

(D) none of these

Ans. Option (A) is correct.

Explanation: Given that,

$f(x) = e^{|x|}$

The functions e^x and $|x|$ are continuous functions for all real value of x.

Since e^x is differentiable everywhere but $|x|$ is non-differentiable at $x = 0$.

Thus, the given function $f(x) = e^{|x|}$ is continuous everywhere but not differentiable at $x = 0$.

Q. 10. Let $f(x) = |\sin x|$, then:

(A) f is everywhere differentiable

(B) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

(C) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$.

(D) none of these

Ans. Option (B) is correct.

Explanation: Given that,

$$f(x) = |\sin x|$$

The functions $|x|$ and $\sin x$ are continuous function for all real value of x.

Thus, the function $f(x) = |\sin x|$ is continuous function everywhere.

Now, $|x|$ is non-differentiable function at $x = 0$.

Since $f(x) = |\sin x|$ is non-differentiable function at $\sin x = 0$

Thus, f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

Q. 11. If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to:

(A) $\frac{4x^3}{1-x^4}$ **(B)** $\frac{-4x}{1-x^4}$

(C) $\frac{1}{4-x^4}$ **(D)** $\frac{-4x^3}{1-x^4}$

Ans. Option (B) is correct.

Explanation: Given that,

$$y = \log\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad y = \log(1-x^2) - \log(1+x^2).$$

Differentiate with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(1-x^2)\right] - \frac{d}{dx}\left[\log(1+x^2)\right]$$

$$= \frac{-2x}{1-x^2} - \frac{2x}{1+x^2}$$

$$= -2x\left(\frac{2}{(1-x^2)(1+x^2)}\right)$$

$$= -\frac{4x}{1-x^4}$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Let $f(x) = x|x|$, for all $x \in$ R check its differentiability at $x = 0$.

R&U [CBSE Delhi Set-III 2020]

Sol. $f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0, \end{cases}$, differentiable at $x = 0$. ½+½

[CBSE Marking Scheme 2020]

Detailed Solution:

Here,

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

$$R.f'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{x^2 - 0}{x - 0} = 0$$

$$L.f'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{-x^2 - 0}{x - 0} = 0$$

$$R.f'(0) = L.f'(0)$$

Therefore $f(x)$ is differentiable at $x = 0$.

Commonly Made Error

- Mostly students find difficulty in differentiating modulus functions.

Answering Tip

- Learn the modulus function with its properties.

Q. 2. If $y = x|x|$, find $\frac{dy}{dx}$ for $x < 0$.

R [CBSE OD Set-I, 2019]

Q. 3. Differentiate $e^{\sqrt{3x}}$, with respect to x.

R [CBSE OD Set-II, 2019]

Short Answer Type Questions-I (2 marks each)

Q. 1. If $y = ae^{2x} + be^{-x}$, then show that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.

[CBSE SQP-2020-21]

Q. 2. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$, $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$.

A I R&U [CBSE OD Set I, II, III-2020]

Sol. $\dfrac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta$ ½

$\dfrac{dy}{d\theta} = \cos\theta - 2\cos 2\theta$ ½

$\therefore \dfrac{dy}{dx} = \dfrac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$ ½

$\therefore \left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \sqrt{3}$ ½

[CBSE Marking Scheme 2020]

Detailed Solution:

Given, $x = \cos\theta - 2\cos 2\theta$

$y = \sin\theta - \sin 2\theta$

Differentiate $x = \cos\theta - 2\cos 2\theta$ w.r.t θ

$$\frac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta \quad \text{...(i)}$$

Differentiate $y = \sin\theta - \sin 2\theta$ w.r.t θ

$$\frac{dy}{d\theta} = \cos\theta - \cos 2\theta \quad \text{...(ii)}$$

On dividing eq(ii) by eq(i)

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} = \frac{\cos\left(\frac{\pi}{3}\right) - 2\cos 2\left(\frac{\pi}{3}\right)}{2\sin 2\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{\left(\frac{1}{2}\right) - 2\left(\frac{-1}{2}\right)}{2\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{\frac{1}{2}+1}{\frac{\sqrt{3}}{2}} = \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \sqrt{3}$$

Commonly Made Error

- Some students first substitute the value and then take the derivative which is wrong.

Answering Tip

- Find the derivative first and then substitute the values.

Q. 3. If $x = a\cos\theta$; $y = b\sin\theta$, then find $\dfrac{d^2y}{dx^2}$.

[CBSE Delhi Set-I,II,III 2020]

Q. 4. Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$.

[CBSE Delhi Set I, II, III-2020]

Sol. Let $y = \sin^2 x$ and $z = e^{\cos x}$

$\therefore \dfrac{dy}{dx} = 2\sin x\cos x$

and $\dfrac{dz}{dx} = -\sin x.e^{\cos x}$ ½+½

$\therefore \dfrac{dy}{dz} = \dfrac{2\sin x\cos x}{-\sin x\, e^{\cos x}}$

$= \dfrac{-2\cos x}{e^{\cos x}}$ or $-2\cos x\, e^{-\cos x}$

½+½

[CBSE Marking Scheme 2020]

Detailed Solution

Here, suppose $u = \sin^2 x,\ v = e^{\cos x}$

Then, we need to differentiate u w.r. to v

i.e. $\dfrac{du}{dv} = \dfrac{du}{dx}\cdot\dfrac{dx}{dv}$

$$= \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{d(\sin^2 x)}{dx}}{\frac{d(e^{\cos x})}{dx}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2\sin x\cos x}{e^{\cos x}.(-\sin x)} = -\frac{2\cos x}{e^{\cos x}}$$

Q. 5. Find the derivative of $x^{\log x}$ w.r.t. $\log x$.

[CBSE OD Set-II 2020]

Sol. Let $u = x^{\log x}$ and $v = \log x$

Now, $\log u = (\log x)^2$

$\Rightarrow \dfrac{1}{u}\dfrac{du}{dx} = 2\log x.\dfrac{1}{x}$

$\Rightarrow \dfrac{du}{dx} = \dfrac{2\log x}{x}.x^{\log x}$ 1

Again, $v = \log x$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{1}{x}$ ½

$\therefore \dfrac{du}{dv} = 2.x^{\log x}\log x$ ½

[CBSE Marking Scheme 2020]v

Detailed Solution:

Let $u = x^{\log x}$, $v = \log x$ and $u = x^{\log x}$

taking log on both sides

$\log u = \log x^{\log x}$

$\log u = \log x \log x$

$\log u = (\log x)^2$

Differentiate w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = 2\log x \times \frac{1}{x}$$

$$\frac{du}{dx} = u \cdot \frac{2\log x}{x}$$

$$\frac{du}{dx} = \frac{x^{\log x} . 2\log x}{x} \quad ...(i)$$

$$v = \log x$$

Differentiate w.r.t. to x

$$\frac{dv}{dx} = \frac{1}{x} \quad ...(ii)$$

On dividing eq(i) by eqn. (ii)

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{x^{\log x} . \frac{2\log x}{x}}{\frac{1}{x}}$$

$$\frac{du}{dv} = 2\log x . x^{\log x}$$

The derivative of $x^{\log x}$ w.r.t. to $\log x$ is $2\log x . x^{\log x}$.

Q. 6. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

R&U [Delhi/OD, 2018]

Q. 7. Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

R&U [Delhi Set I, II 2017]

Sol. From the given equation

$$2\sin y \cos y . \frac{dy}{dx} - \sin xy . \left[x . \frac{dy}{dx} + y.1\right] = 0 \quad \mathbf{1}$$

or
$$\frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x\sin(xy)}$$

$$\frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x \sin xy}$$

$$\left(\frac{dy}{dx}\right)_{x=1, y=\frac{\pi}{4}} = \frac{\frac{\pi}{4}\sin 1 . \frac{\pi}{4}}{\sin 2\frac{\pi}{4} - 1\sin 1 . \frac{\pi}{4}} = \frac{\frac{\pi}{4} . \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{4(\sqrt{2}-1)}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=1, y=\frac{\pi}{4}} = \frac{\pi}{4\left(\sqrt{2}-1\right)} \quad \mathbf{1}$$

[CBSE Marking Scheme, 2017]

Q. 8. If $y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right), -\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$.

R&U [Delhi Set III, 2017]

Sol.
$$y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right), -\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$$

put $3x = \sin\theta$ or $\theta = \sin^{-1} 3x$ ½

$$y = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\sin^{-1} 3x \quad ½$$

$$\therefore \quad \frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}} \quad \mathbf{1}$$

[CBSE Marking Scheme, 2017]

Q. 9. Differentiate $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ with respect to x.

R&U [Comptt. Set I, II, III, 2018]

Sol. Let
$$y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \tan^{-1}\left[\frac{\cos x\left(1 - \frac{\sin x}{\cos x}\right)}{\cos x\left(1 + \frac{\sin x}{\cos x}\right)}\right] \quad ½$$

$$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) \quad ½$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right] \quad ½$$

$$= \frac{\pi}{4} - x$$

$$\Rightarrow \quad \frac{dy}{dx} = -1 \quad ½$$

[CBSE Marking Scheme, 2018]

Short Answer Type Questions-II (3 marks each)

Q. 1. If $y = e^{x\sin^2 x} + (\sin x)^x$, find $\frac{dy}{dx}$.

A I R&U [CBSE SQP 2020-21]

Sol. Let $u = e^{x\sin^2 x}$

and $v = (\sin x)^x$ ½

so that $y = u + v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

Now, $u = e^{x\sin^2 x}$,

Differentiating both sides w.r.t. x, we get

$$\Rightarrow \quad \frac{du}{dx} = e^{x\sin^2 x}[x(\sin 2x) + \sin^2 x] \quad \mathbf{1}$$

...(2)

Also, $v = (\sin x)^x$

$\Rightarrow \log v = x \log(\sin x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = x\cot x + \log(\sin x)$$

$$\frac{dv}{dx} = (\sin x)^x [x \cot x + \log(\sin x)]$$

...(3) **1**

Substituting from eq.(2), eq.(3) in eq. (1) we get

$$\frac{dy}{dx} = e^{x\sin^2 x}[\sin 2x + \sin^2 x] + (\sin x)^x [x \cot x + \log(\sin x)]$$ ½

[CBSE SQP Marking Scheme 2020-21]

Commonly Made Error

- Some students take logarithms directly without splitting the function.

Answering Tip

- There is no formula to find log $(a + b)$.

Q. 2. Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x = 1$.

R&U [CBSE SQP 2020-21]
[CBSE Board 2021]

Q. 3. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

U [NCERT Exemplar]
[CBSE SQP-2020-21]

Sol. $$\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$$

Let $x = \sin A$, $y = \sin B$

$$\sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$$ ½

$$\cos A + \cos B = a(\sin A - \sin B)$$

$$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2a\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$ **1**

$$\Rightarrow \cos\left(\frac{A-B}{2}\right) = a\sin\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \cot\left(\frac{A-B}{2}\right) = a$$

$$\Rightarrow \frac{A-B}{2} = \cot^{-1} a$$ **1**

$$A - B = 2\cot^{-1} a$$

$$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$$

Differentiating w.r.t. x

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$ **1**

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$ ½

[CBSE SQP Marking Scheme, 2020] (Modified)

Commonly Made Error

- Most students differentiate directly and will fail to reach the final answer.

Answering Tip

- Learn all the substitutions made in inverse trigonometric functions by heart.

Q. 4. If $x = a\sec\theta$, $y = b\tan\theta$. Find $\frac{d^2y}{dx^2}$ at $x = \frac{\pi}{6}$.

R&U [CBSE SQP 2020-21]

Q. 5. If $x = a(\cos 2\theta + 2\theta \sin 2\theta)$ and $y = a(\sin 2\theta - 2\theta\cos 2\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{8}$.

A I [CBSE SQP-2020-21]

Sol. $$x = a(\cos 2\theta + 2\theta\sin 2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a(-2\sin 2\theta + 2\sin 2\theta + 4\theta\cos 2\theta)$$ ½

$$\Rightarrow \frac{dx}{d\theta} = a(4\theta\cos 2\theta)$$ ...(1)

$$y = a(\sin 2\theta - 2\theta\cos 2\theta)$$

$$\Rightarrow \frac{dy}{d\theta} = a(2\cos 2\theta + 4\theta\sin 2\theta - 2\cos 2\theta)$$ ½

$$\Rightarrow \frac{dy}{d\theta} = a(4\theta\sin 2\theta)$$ ...(2)

Using (1) and (2),

$$\Rightarrow \frac{dy}{dx} = \frac{a(4\theta\sin 2\theta)}{a(4\theta\cos 2\theta)}$$ ½

$$\Rightarrow \frac{dy}{dx} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

Differentiating again with respect to x, we get

$$\Rightarrow \frac{d^2y}{dx^2} = 2\sec^2 2\theta.\frac{d\theta}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\sec^2 2\theta . \frac{1}{a(4\theta \cos 2\theta)} \quad \text{½}$$

$$\Rightarrow \quad \left.\frac{d^2y}{dx^2}\right]_{\theta = \frac{\pi}{8}} = 2\sec^2 \frac{\pi}{4} \cdot \frac{1}{a\left(4\,\frac{\pi}{8}\cos\frac{\pi}{4}\right)}$$

$$= \frac{8\sqrt{2}}{\pi a} \quad \mathbf{1}$$

[CBSE SQP Marking Scheme, 2020] (Modified)

Q. 6. If $x = a\sec^3\theta$, $y = a\tan^3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$. **[CBSE Delhi Set-II, 2020]**

Topper Answer, 2020

Sol. $x = a\sec^3\theta$

Differentiating with respect to x

$\Rightarrow \frac{dx}{d\theta} = 3a\sec^2\theta\sec\theta\tan\theta = 3a\sec^3\theta\tan\theta$.

$y = a\tan^3\theta$

Differentiating with respect to y

$\Rightarrow \frac{dy}{d\theta} = 3a\tan^2\theta\sec^2\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$. —(i)

Differenting equation (i) with respect to x

$\frac{d^2y}{dx^2} = \frac{d\sin\theta}{d\theta}\frac{d\theta}{dx}$

$\frac{d^2y}{dx^2} = \frac{\cos\theta}{3a\sec^3\theta\tan\theta} = \frac{\cos\theta\cos^3\theta\cos\theta}{3a\sin\theta} = \frac{\cos^5\theta}{3a\sin\theta}$

$\left(\frac{d^2y}{dx^2}\right)_{(\theta=\pi/4)} = \frac{\sqrt{2}}{(\sqrt{2})^5 3a} = \frac{1}{12a}$ Answer

Q. 7. If $y = \sin^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right)$, then show that $\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$.

A **[CBSE OD Set I, II, III-2020]**

Sol. Put $x = \cos 2\theta$

$$\Rightarrow \quad \theta = \frac{1}{2}\cos^{-1}x \quad \mathbf{1}$$

$$\therefore \quad y = \sin^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{2}\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{4}+\theta\right)\right)$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x \quad \text{½}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}} \quad \text{½}$$

[CBSE Marking Scheme 2020] (Modified)

Detailed Solution:

To Prove: $\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$

Given: $y = \sin^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right)$

Put $x = \cos 2\theta$

$$y = \sin^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{2}\right]$$

$$y = \sin^{-1}\left[\frac{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-(1-2\sin^2\theta)}}{2}\right]$$

[$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$]

$$y = \sin^{-1}\left[\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{2}\right]$$

$$y = \sin^{-1}\left[\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right]$$

$$y = \sin^{-1}\left[\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta\right]$$

[Using $\sin A \cos B + \cos A \sin B = \sin(A + B)$]

$$y = \sin^{-1}\left[\sin\left(\frac{\pi}{4}+\theta\right)\right]$$

$$y = \frac{\pi}{4}+\theta$$

Put $\theta = \frac{1}{2}\cos^{-1}x$ [$x = \cos 2\theta$]

$$y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x$$

Differentiate above equation w.r.t x :

$$\frac{dy}{dx} = 0+\frac{1}{2}\times\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$$ **Hence Proved.**

Commonly Made Error

- Some students directly apply the formula for derivative of $\sin^{-1}x$ and apply chain rule.

Answering Tip

- Convert inverse trigonometric functions to simplest form before finding the derivatives.

Q. 8. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ and $x \neq y$ prove that $\frac{dy}{dx}=-\frac{1}{(x+1)^2}$.

A I R [CBSE OD Set-I, 2019][NCERT]

Q. 9. If $x^y - y^x = a^b$, find $\frac{dy}{dx}$.

U [CBSE Delhi Set-III, 2019]

Sol. Let $u = x^y$, $v = y^x$. Then $u - v = a^b$ ...(1)

$$\Rightarrow \frac{du}{dx}-\frac{dv}{dx}=0$$ ½

Now, $\log u = y.\log x$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx}=y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx} \Rightarrow \frac{du}{dx}=x^y\left(\frac{y}{x}+\log x\cdot\frac{dy}{dx}\right)$$ ½

Again, $\log v = x \cdot \log y$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx}=x\cdot\frac{1}{y}\cdot\frac{dy}{dx}+\log y\cdot 1 \Rightarrow \frac{dv}{dx}=y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)$$ 1

From (1), (2) and (3)

$$x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)-y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)=0$$ ½

$$\Rightarrow \frac{dy}{dx}=\frac{y^x\cdot\log y-x^{y-1}\cdot y}{x^y\cdot\log x-y^{x-1}\cdot x}$$ ½

[CBSE Marking Scheme, 2019] (Modified)

Detailed Solution:

Given, $x^y - y^x = a^b$

$\Rightarrow e^{y\log x} - e^{x\log y} = e^{b\log a}$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(e^{y\log x})-\frac{d}{dx}(e^{x\log y})=\frac{d}{dx}(e^{b\log a})$$

$$\Rightarrow e^{y\log x}\frac{d}{dx}(y\log x)-e^{x\log y}\frac{d}{dx}(x\log y)=0$$

$$\Rightarrow x^y\left\{\frac{dy}{dx}\times\log x+y\times\frac{1}{x}\right\}-y^x\left\{1\times\log y+x\times\frac{1}{y}\times\frac{dy}{dx}\right\}=0$$

$$\Rightarrow \left\{x^y\times\log x-y^x\times\frac{x}{y}\right\}\frac{dy}{dx}+\left\{x^y\times\frac{y}{x}-y^x\times\log y\right\}=0$$

$$\Rightarrow \{x^y\log x-xy^{x-1}\}\frac{dy}{dx}+\{yx^{y-1}-y^x\log y\}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{\{y^x\log y-x^{y-1}y\}}{\{x^y\log x-y^{x-1}x\}}$$

Commonly Made Error

- Some students take logarithm directly without separating the equation.

Answering Tip

- $\log(a + b) \neq \log a + \log b$. So separate the function and find the derivatives.

Q. 10. If $y = (\sin^{-1}x)^2$, prove that $(1 - x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0$.

A I R [CBSE Delhi Set I-2019]

Q. 11. If $(\cos x)^y = (\sin y)^x$ find $\frac{dy}{dx}$.

R [CBSE OD Set-I, 2019]

Sol. $(\cos x)^y = (\sin y)^x \Rightarrow y \cdot \log(\cos x) = x \cdot \log(\sin y)$ **1**

Differentiating w.r.t 'x'

$$\Rightarrow \frac{dy}{dx}\cdot\log(\cos x) + y(-\tan x) \quad \mathbf{1}$$

$$= \log(\sin y) + x\cdot\cot y\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\cdot\tan x + \log(\sin y)}{\log(\cos x) - x\cot y} \quad \mathbf{1}$$

[CBSE Marking Scheme, 2019] (Modified)

Detailed Solution:

Given, $(\cos x)^y = (\sin y)^x$

Taking log on both sides, we get

$$y\log(\cos x) = x\log(\sin y)$$

Differentiating w.r.t x, we get

$$\Rightarrow y\frac{d}{dx}[\log(\cos x)] + \log(\cos x)\frac{d}{dx}(y)$$

$$= x\frac{d}{dx}[\log(\sin y)] + \log(\sin y)\frac{d}{dx}(x)$$

$$\Rightarrow y\left(\frac{-\sin x}{\cos x}\right) + \log(\cos x)\frac{dy}{dx}$$

$$= x\left(\frac{\cos y}{\sin y}\right)\frac{dy}{dx} + \log(\sin y).1$$

$$\Rightarrow -y\tan x + \log(\cos x)\frac{dy}{dx} = x\cot y\frac{dy}{dx} + \log(\sin y)$$

$$\Rightarrow (\log(\cos x) - x\cot y)\frac{dy}{dx} = \log(\sin y) + y\tan x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log(\sin y) + y\tan x}{\log(\cos x) - x\cot y}$$

Q. 12. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, Prove that $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ is a constant independent of a and b.

R [NCERT] [CBSE OD Set-I, 2019]

Sol. $(x-a)^2 + (y-b)^2 = c^2,\ c > 0$

$$2(x-a) + 2(y-b)\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x-a}{y-b} \quad \mathbf{1}$$

$$\frac{d^2y}{dx^2} = -\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}$$

$$= \frac{-c^2}{(y-b)^3}\left(\text{By substitution } '\frac{dy}{dx}'\right) \quad \tfrac{1}{2}$$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{3/2}}{-\frac{c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}} = -c \quad 1$$

which is a constant independent of 'a' and 'b'. ½

[CBSE Marking Scheme, 2019] (Modified)

Detailed Solution:

Given, $(x-a)^2 + (y-b)^2 = c^2$ ...(i)

Differentiating w.r.t. x, we get

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow (x-a) + (y-b)\frac{dy}{dx} = 0 \quad \text{...(ii)}$$

Differentiating again w.r.t. x, we get

$$1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow (y-b)\frac{d^2y}{dx^2} = -\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{y-b} \quad \text{...(iii)}$$

From eq (ii), we have

$$\frac{dy}{dx} = -\left(\frac{x-a}{y-b}\right)$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x-a)^2}{(y-b)^2}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \frac{(x-a)^2+(y-b)^2}{(y-b)^2}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \frac{c^2}{(y-b)^2} \text{ [from eq (i)] ...(iv)}$$

$$\Rightarrow \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2} = \left\{\frac{c^2}{(y-b)^2}\right\}^{3/2} = \frac{c^3}{(y-b)^3} \quad \text{...(v)}$$

From eqns. (iii) and (iv), we get

$$\frac{d^2y}{dx^2} = -\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{y-b} = -\frac{c^2/(y-b)^2}{(y-b)}$$

$$= \frac{-c^2}{(y-b)^3} \quad \text{...(vi)}$$

From eqns. (v) and (vi), we obtain

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\frac{c^3}{(y-b)^3}}{\frac{-c^2}{(y-b)^3}} = -c,$$ which is independent of a and b.

Hence Proved.

Q. 13. If $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

U [CBSE Delhi Set-III, 2019]

Sol. $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

Differentiating both sides w.r.t. x,

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right) = 2\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{x\cdot\frac{dy}{dx}-y\cdot 1}{x^2}\right) \quad \mathbf{2}$$

$$\Rightarrow \frac{2}{x^2+y^2}\left(x+y\frac{dy}{dx}\right) = \frac{2x^2}{x^2+y^2}\cdot\frac{1}{x^2}\cdot\left(x\frac{dy}{dx}-y\right) \quad \mathbf{1}$$

$$\Rightarrow (x+y) = (x-y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \quad \mathbf{1}$$

[CBSE Marking Scheme, 2019]

Detailed Solution:

Topper Answer, 2019

Sol. Q14 $\log(x^2+y^2) = 2\tan^{-1}\frac{y}{x}$.

Diff. both sides w.r.t x

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right) = \frac{2}{1+\frac{y^2}{x^2}}\cdot\frac{d}{dx}\left(\frac{y}{x}\right)$$

$\left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2};\ \frac{d}{dx}(\log x) = \frac{1}{x}\right]$

$$\Rightarrow \frac{x+yy'}{x^2+y^2} = \frac{x^2}{x^2+y^2}\cdot\frac{(xy'-y)}{x^2}$$

[where $y' = dy/dx$]

$\Rightarrow x + yy' = xy' - y$

$\Rightarrow x + y = y'(x-y)$

$\Rightarrow y' = \frac{x+y}{x-y}$

$$\boxed{\frac{dy}{dx} = \frac{x+y}{x-y}}$$

Hence proved.

Q. 14. If $x = \cos t + \log\tan\frac{t}{2}$, $y = \sin t$, then find the value of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

A I R&U [CBSE OD Set I, II, III-2019]

Q. 15. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

R&U [Delhi/OD, 2018]

Long Answer Type Questions (5 marks each)

Q. 1. Find $\frac{dy}{dx}$, if $y = e^{\sin^2 x}\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$.

A [S.Q.P. Dec. 2016-17] [HOTS]

Sol. Putting $x = \cos 2\theta$ in $\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$, we get **1**

$$2\tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$$

i.e., $2\tan^{-1}\sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}}$

$= 2\tan^{-1}(\tan\theta)$

$= 2\theta = \cos^{-1}x$ **1**

Hence, $y = e^{\sin^2 x}\cos^{-1}x$

or $\log y = \sin^2 x + \log(\cos^{-1}x)$

or $\dfrac{1}{y}\times\dfrac{dy}{dx} = 2\sin x\cos x + \dfrac{1}{\cos^{-1}x}\times\dfrac{-1}{\sqrt{1-x^2}}$

$= \sin 2x - \dfrac{1}{\cos^{-1}x\sqrt{1-x^2}}$ **2**

or $\dfrac{dy}{dx} = e^{\sin^2 x}\cos^{-1}x\left[\sin 2x - \dfrac{1}{\cos^{-1}x\sqrt{1-x^2}}\right]$ **1**

[CBSE Marking Scheme, 2016] (Modified)

Q. 2. If $x\cos(a + y) = \cos y$, then prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$. Hence, show that $\sin a\dfrac{d^2y}{dx^2} + \sin 2(a+y)\dfrac{dy}{dx} = 0$.

COMPETENCY BASED QUESTIONS

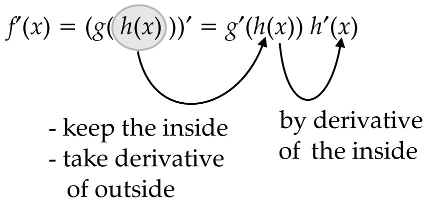
Case based MCQs (4 marks each)

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

Ms. Remka of city school is teaching chain rule to her students with the help of a flow-chart

The chain rule says that if h and g are functions and $f(x) = g(h(x))$, then



Q. 1. Find $\dfrac{d}{dx}(\cos x^5)$.

(A) $x^4\sin x^5$ **(B)** $-5x^4\sin x^5$

(C) $5x^4\sin x^5$ **(D)** $4x^5\sin x^4$

Ans. Option (B) is correct.

Explanation: $\dfrac{d}{dx}(\cos x^5) = -\sin x^5\dfrac{d}{dx}(x^5)$

$= -\sin x^5(5x^4)$

$= -5x^4\sin x^5$

Q. 2. Find $\dfrac{d}{dx}\sin(\cos x)$.

(A) $\cos(\cos x)$ **(B)** $\sin x\cos(\cos x)$

(C) $-\sin x.\cos(\cos x)$ **(D)** $\cos x\sin(\cos x)$

Ans. Option (C) is correct.

Explanation: $\dfrac{d}{dx}\sin(\cos x) = \cos(\cos x)\dfrac{d}{dx}(\cos x)$

$= \cos(\cos x)(-\sin x)$

$= -\sin x.\cos(\cos x)$

Q. 3. $\dfrac{d}{dx}(\sin^3 x) =$ ______.

(A) $\cos^3 x$ **(B)** $3\sin x\cos x$

(C) $3\sin^2 x\cos x$ **(D)** $-\cos^3 x$

Ans. Option (C) is correct.

Explanation:

$$\frac{d}{dx}(\sin^3 x) = 3\sin^2 x\frac{d}{dx}(\sin x)$$

$$= 3\sin^2 x\cos x$$

Q. 4. $\dfrac{d}{dx}\sin x^3$ ______.

(A) $\cos(x^3)$ **(B)** $-\cos(x^3)$

(C) $3x^2\sin(x^3)$ **(D)** $3x^2\cos(x^3)$

Ans. Option (D) is correct.

Explanation:

$$\frac{d}{dx}(\sin x^3) = \cos x^3\frac{d}{dx}(x^3)$$

$$= 3x^2\cos x^3$$

Q. 5. $\dfrac{d}{dx}(\sin 2x)$ at $x = \dfrac{\pi}{2}$ is ______.

(A) 0 **(B)** 1

(C) 2 **(D)** –2

Ans. Option (D) is correct.

Explanation:

$$\frac{d}{dx}(\sin 2x) = \cos 2x\frac{d}{dx}(2x)$$

$$= 2\cos 2x$$

At $x = \dfrac{\pi}{2}$, $\dfrac{d}{dx}(\sin 2x) = 2\cos\pi = -2$

II. Read the following text and answer the following questions on the basis of the same:

A potter made a mud vessel, where the shape of the pot is based on $f(x) = |x - 3| + |x - 2|$, where $f(x)$ represents the height of the pot. **[CBSE QB-2021]**

Q. 1. When $x > 4$ what will be the height in terms of x?

(A) $x - 2$ **(B)** $x - 3$

(C) $2x - 5$ **(D)** $5 - 2x$

Ans. Option (C) is correct.

Explanation: The given function can be written as

$$f(x) = \begin{cases} 5-2x, & \text{if } x < 2 \\ 1, & \text{if } 2 \le x < 3 \\ 2x-5, & \text{if } x \ge 3 \end{cases}$$

When $x > 4$, $f(x) = 2x - 5$

Q. 2. Will the slope vary with x value?

(A) Yes **(B)** No

(C) Can't say **(D)** Incomplete data

Ans. Option (A) is correct.

Explanation:

$$f'(x) = \begin{cases} -2, & \text{if } x < 2 \\ 0, & \text{if } 2 \le x < 3 \\ 2, & \text{if } x \ge 3 \end{cases}$$

Q. 3. What is $\frac{dy}{dx}$ at $x = 3$:

(A) 2

(B) –2

(C) Function is not differentiable

(D) 1

Ans. Option (C) is correct.

Explanation: $f(x)$ is not differentiable at $x = 2$ and $x = 3$.

Q. 4. When the value of x lies between (2, 3) then the function is:

(A) $2x - 5$ **(B)** $5 - 2x$

(C) 1 **(D)** 5

Ans. Option (C) is correct.

Explanation: In (2, 3), $f(x) = 1$

Q. 5. If the potter is trying to make a pot using the function $f(x) = [x]$, will he get a pot or not? Why?

(A) Yes, because it is a continuous function

(B) Yes, because it is not continuous

(C) No, because it is a continuous function

(D) No, because it is not continuous

[CBSE QB 2021]

Ans. Option (D) is correct.

Explanation: $[x]$ is not continuous at integral values of x.

Case based Subjective Questions (4 marks each)

I. Read the following text and answer the following questions on the basis of the same:

(Each Sub-part carries 2 marks)

Reena started to read the notes on the topic 'differentiability' which she has prepared in the class of mathematics. She wanted to solve the questions based on this topic, which teacher gave as home work. She has written following matter in her notes :

Let $f(x)$ be a real valued function, then its Left Hand derivative (LHD) is :

$$Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

Right Hand Derivative (RHD) is :

$$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its LHD and RHD at $x = a$ exist and one equal.

For the function, $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$

Q. 1. Check the differentiability of function at $x = 1$.

Sol. We have, $f(x) = \begin{cases} x-3, & x \ge 3 \\ 3-x, & 1 \le x < 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$

$$\text{LHD at } x = 1 = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{-1}{h}\left[\frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4} - 2\right]$$

$$= \lim_{h \to 0} \left(\frac{1 + h^2 - 2h - 6 + 6h + 13 - 8}{-4h}\right)$$

$$= \lim_{h \to 0} \left(\frac{h^2 + 4h}{-4h}\right) = 1$$ **1**

$$\text{RHD at } x = 1 = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{3 - (1+h) - 2}{h}$$

$$= \lim_{h \to 0} \frac{3 - (1+h) - 2}{h}$$

$$= \lim_{h \to 0} -\frac{h}{h} = -1$$

Since, LHD = RHD

Thus, $f(x)$ is differentiable at $x = 1$. **1**

Q. 2. Find the value of $f'(-1)$ and $f(2)$.

Sol. $f'(x) = \frac{x}{2} - \frac{3}{2}, x < 1$

$\therefore \quad f'(-1) = \frac{-1}{2} - \frac{3}{2} = -2$ **1**

$f'(x) = -1, 1 \le x \le 3$

$\therefore \quad f'(2) = -1$ **1**

II. Read the following text and answer the following questions on the basis of the same:

If a relation between x and y is such that y cannot be expressed in terms of x, then y is called implicit function of x. When a given relation expresses y as an implicit function of x and we want to find $\frac{dy}{dx}$.

Then we differentiate every term of the given relation w.r.t. x remembering that a term in y is first differentiated w.r.t. y and then multiplied by $\frac{dy}{dx}$.

Q. 1. If $x^3 + x^2y + xy^2 + y^3 = 81$, then find $\frac{dy}{dx}$.

Sol. We have,

$$x^3 + x^2y + xy^2 + y^3 = 81$$

Differentiating both sides w.r.t. x, we get

$$3x^2 + \left(2xy + x^2\frac{dy}{dx}\right) + \left(y^2 + 2xy\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (x^2 + 2xy + 3y^2)\frac{dy}{dx} + (3x^2 + 2xy + y^2) = 0$$ **1**

$$\Rightarrow (x^2 + 2xy + 3y^2)\frac{dy}{dx} = -(3x^2 + 2xy + y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}$$ **1**

Q. 2. If $y = \left(\sqrt{x}\right)^y$, then find $\frac{dy}{dx}$.

Sol. We have,

$$y = \left(\sqrt{x}\right)^y$$

$$\Rightarrow \log y = y\log\left(\sqrt{x}\right)$$

[Taking by both sides]

Now, differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\frac{d}{dx}\left(\log\sqrt{x}\right) + \log\sqrt{x}\frac{d}{dx}(y)$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = y\left(\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\right) + \log\sqrt{x}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{2x} + \frac{dy}{dx}\left(\frac{1}{2}\log x\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - \frac{1}{2}\log x\right) = \frac{y}{2x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x} \times \frac{2y}{2 - y\log x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(2 - y\log x)}$$

Solutions for Practice Questions (Topic-1)

Very Short Answer Type Questions

1. $\lim_{x\to3}\frac{x^2-9}{x-3} = 6, \quad \therefore k = 6$ ½ + ½

[CBSE Marking Scheme 2020]

Commonly Made Error

- Some students do not know how to evaluate limits of the form $\frac{0}{0}$.

Answering Tip

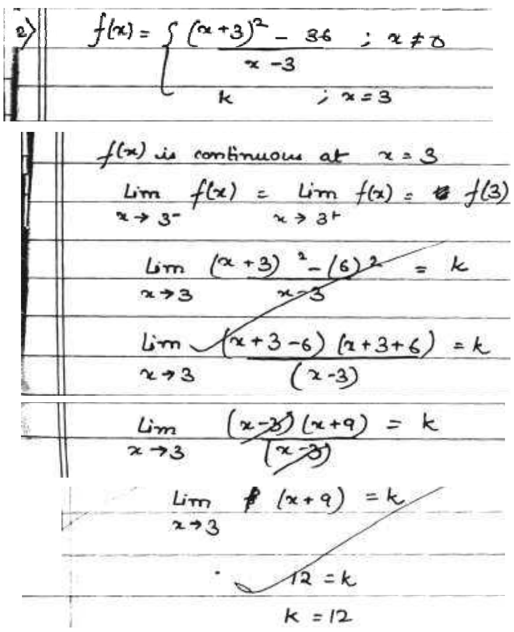
- From this part questions are mainly asked from indeterminate form of limits. Learn all the different forms of limits.

2. $x = 12$ **[CBSE Marking Scheme 2017] 1**

Detailed Solution:

Topper Answer, 2017

Sol.


2) $f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3} & ; x \ne 3 \\ k & ; x = 3 \end{cases}$

$f(x)$ is continuous at $x = 3$

$\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x) = f(3)$

$\lim_{x\to3} \frac{(x+3)^2 - (6)^2}{x-3} = k$

$\lim_{x\to3} \frac{(x+3-6)(x+3+6)}{(x-3)} = k$

$\lim_{x\to3} \frac{(x-3)(x+9)}{(x-3)} = k$

$\lim_{x\to3} (x+9) = k$

$12 = k$

$k = 12$

Short Answer Type Questions-I

2. $\lim_{x\to 2} f(x) = f(2) = k$ ½

$\lim_{x\to 2}\frac{(x+5)(x-2)}{x-2} = k$ $(x \neq 2)$ **1**

$\therefore$ $k = 7$ ½

[CBSE Marking Scheme, 2017]

4. Here, LHL $= \lim_{x\to 0^-} k\cdot\sin\frac{\pi}{2}(x+1) = k$ **1**

or RHL $= \lim_{x\to 0^+}\frac{\tan x(1-\cos x)}{x^3}$ **1**

$= \lim_{x\to 0^+}\frac{\tan x}{x}.2\left(\frac{\sin\frac{x}{2}}{2\frac{x}{2}}\right)^2$ ½

$= 1\times 2\times\frac{1}{4}$

$\Rightarrow$ $k = \frac{1}{2}$ ½

[CBSE Marking Scheme, 2016] (Modified)

Solutions for Practice Questions (Topic-2)

Very Short Answer Type Questions

2. For $x < 0, y = x|x| = -x^2$ ½

$\therefore$ $\frac{dy}{dx} = -2x$ ½

[CBSE Marking Scheme, 2019]

Detailed Solution:

Given, $y = x|x|$

Since, $x < 0$, then $|x| = -x$

$\therefore$ $y = x(-x)$

$y = -x^2$

$\therefore$ $\frac{dy}{dx} = -2x$

3. $\frac{d}{dx}\left(e^{\sqrt{3x}}\right) = \frac{\sqrt{3}}{2\sqrt{x}}e^{\sqrt{3x}}$ **1**

[CBSE Marking Scheme, 2019]

Detailed Solution:

Let $y = e^{\sqrt{3x}}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}(e^{\sqrt{3x}})$

$\Rightarrow$ $\frac{dy}{dx} = e^{\sqrt{3x}}\frac{d}{dx}\sqrt{3x}$

$\Rightarrow$ $\frac{dy}{dx} = e^{\sqrt{3x}}\sqrt{3}\frac{d}{dx}(\sqrt{x})$

$\Rightarrow$ $\frac{dy}{dx} = \sqrt{3}e^{\sqrt{3x}}\frac{1}{2\sqrt{x}}\frac{d}{dx}(x)$

$\Rightarrow$ $\frac{dy}{dx} = \frac{\sqrt{3}}{2}e^{\sqrt{3x}}.\frac{1}{\sqrt{x}}.1$

$\Rightarrow$ $\frac{dy}{dx} = \frac{\sqrt{3}}{2\sqrt{x}}e^{\sqrt{3x}}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{\sqrt{3x}}{2x}e^{\sqrt{3x}}$

Commonly Made Error

- Sometimes students forget the Chain Rule.

Answering Tip

- A thorough revision of the formula is must.

Short Answer Type Questions-I

1. $y = ae^{2x} + be^{-x}$ ... (1)

$\frac{dy}{dx} = 2ae^{2x} - be^{-x}$ ... (2)

$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$ ... (3) **1**

Putting the values on LHS

$= \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y$

$= (4ae^{2x} + be^{-x}) - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x})$

$= 4ae^{2x} + be^{-x} - 2ae^{2x} + be^{-x} - 2ae^{2x} - 2be^{-x}$

$= 0.$ **1**

[CBSE SQP Marking Scheme, 2020]

3. $\frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta \Rightarrow \frac{dy}{dx} = -\frac{b}{a}\cot\theta$ ½+½

$\frac{d^2y}{dx^2} = \frac{b}{a}\text{cosec}^2\theta\left(\frac{-1}{a\sin\theta}\right) = -\frac{b}{a^2}\text{cosec}^3\theta$ ½+½

[CBSE Marking Scheme 2020]

Detailed Solution:

Here $x = a\cos\theta, y = b\sin\theta$

$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta$

$\Rightarrow \frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{b}{a}\text{cosec}^2\theta \times \frac{d\theta}{dx}$

$= \frac{b}{a}\text{cosec}^2\theta \times \frac{(-1)}{a\sin\theta}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a^2}\text{cosec}^3\theta$

Commonly Made Error

- Mistakes are made while finding second derivative of parametric functions.

Answering Tip

- Second derivative of parametric form is an important area for scoring. Practice more questions from it.

6. $f(x) = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$ **1**

$= \log x^{\log x} = (\log x)^2$ ½

$\therefore f(x) = -\frac{1}{2}$ ½

[CBSE Marking Scheme, 2018]

Commonly Made Error

- Some students write differentiation of $\tan^{-1}x$ directly. They don't simplify the inverse trigonometric functions which leads fail to reach the final answer..

Answering Tip

- Differentiation rules for different functions and terms need attention. A thorough revision is a must.

Short Answer Type Questions-II

2. RHD $= \underset{h\to 0}{Lt}\frac{f(1+h)-f(1)}{h} = \underset{h\to 0}{Lt}\frac{[1+h]-[1]}{h}$

$= \underset{h\to 0}{Lt}\frac{(1-1)}{h} = 0$ **1**

LHD $= \underset{h\to 0}{Lt}\frac{f(1-h)-f(1)}{-h} = \underset{h\to 0}{Lt}\frac{[1-h]-[1]}{-h}$

$= \underset{h\to 0}{Lt}\frac{0-1}{-h}$

$= \underset{h\to 0}{Lt}\frac{1}{h} = \infty$ **1**

Since, RHD $\neq$ LHD

Therefore $f(x)$ is not differentiable at $x = 1$ **1**

[CBSE SQP Marking Scheme 2020-21]

Commonly Made Error

- Mostly students fail to do problems involving modulus and greatest integer functions.

Answering Tip

- Learn the properties of different functions.

4. $y = b\tan\theta$

$\Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$ ...(1)

$x = a\sec\theta$

$\Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$ ...(2)

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\text{cosec}\,\theta$ 1½

Differentiating both sides w.r.t. x, we get

$\frac{d^2y}{dx^2} = \frac{-b}{a}\text{cosec}\,\theta\cot\theta \times \frac{d\theta}{dx}$

$= \frac{-b}{a}\text{cosec}\,\theta\cot\theta \times \frac{1}{a\sec\theta\tan\theta}$ [using (2)]

$= \frac{-b}{a.a}\cot^3\theta$ **1**

$\left.\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{6}} = \frac{-b}{a^2}\left[\cot\frac{\pi}{6}\right]^3$

$= \frac{-b}{a^2}(\sqrt{3})^3 = -\frac{3\sqrt{3}b}{a^2}$ ½

[CBSE SQP Marking Scheme 2020-21]

8. $x\sqrt{1+y}+y\sqrt{1+x}=0 \Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$ ½

Squaring to get: $x^2(1+y)=y^2(1+x)$ ½

Simplifying to get : $(x-y)(x+y+xy)=0$

As, $x \neq y$ $\therefore$ $y=-\dfrac{x}{1+x}$ **1**

$\dfrac{dy}{dx}=\dfrac{-1(1+x)-(-x)\cdot 1}{(1+x)^2}=\dfrac{-1}{(1+x)^2}$ **1**

[CBSE Marking Scheme, 2019] (Modified)

Detailed Solution:

Given, equation is, $x\sqrt{1+y}+y\sqrt{1+x}$ **= 0**

On squaring both sides, we get

$$x^2(1+y)=y^2(1+x)$$
$$\Rightarrow x^2+x^2y=y^2+y^2x$$
$$\Rightarrow x^2-y^2=y^2x-x^2y$$
$$\Rightarrow (x+y)(x-y)=-xy(x-y)$$
$$\Rightarrow (x-y)(x+y)+xy(x-y)=0$$
$$\Rightarrow (x-y)(x+y+xy)=0$$

$\therefore$ Either $x-y=0$ or $x+y+xy=0$

Now, $x-y=0$ or $x=y$

But it is given that $x \neq y$

$\therefore$ $x-y=0$ is rejected.

Now, consider $y+xy+x=0$

or $y(1+x)=-x$

or $y=\dfrac{-x}{1+x}$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{(1+x)\frac{d}{dx}(-x)-(-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(1+x)(-1)+x\times 1}{(1+x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-1-x+x}{(1+x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-1}{(1+x)^2}$$ **Hence Proved.**

10. $y=(\sin^{-1}x)^2$

$\Rightarrow y'=2\cdot\sin^{-1}x\cdot\dfrac{1}{\sqrt{1-x^2}}$ **1**

$\Rightarrow \sqrt{1-x^2}\cdot y'=2\sin^{-1}x$

$\Rightarrow \sqrt{1-x^2}\cdot y''+y'\cdot\dfrac{1}{2\sqrt{1-x^2}}(-2x)=\dfrac{2}{\sqrt{1-x^2}}$ **1**

$\Rightarrow (1-x^2)\cdot y''-xy'=2$ or $(1+x^2)\cdot\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-2=0.$ **1**

[CBSE Marking Scheme, 2019] (Modified)

14. $y=\sin t$

or $\dfrac{dy}{dt}=\cos t$

or $\dfrac{d^2y}{dt^2}=-\sin t$

$\left[\dfrac{d^2y}{dt^2}\right]_{t=\frac{\pi}{4}}=-\sin\dfrac{\pi}{4}=-\dfrac{1}{\sqrt{2}}$

Again, $x=\cos t+\log\tan\dfrac{t}{2}$ **1**

or $\dfrac{dx}{dt}=-\sin t+\dfrac{1}{\tan\frac{t}{2}}\cdot\sec^2\dfrac{t}{2}\cdot\dfrac{1}{2}$

$=-\sin t+\dfrac{\cos\frac{t}{2}}{2\times\sin\frac{t}{2}}\times\dfrac{1}{\cos^2\frac{t}{2}}$

$=-\sin t+\dfrac{1}{\sin 2\times\frac{t}{2}}$

$=-\sin t+\operatorname{cosec} t$ **1**

Now, $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{\cos t}{\operatorname{cosec}t-\sin t}$

$=\dfrac{\cos t}{1-\sin^2 t}.\sin t=\dfrac{\sin t.\cos t}{\cos^2 t}$

$=\tan t$ ½

or $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\dfrac{\sec^2 t}{\operatorname{cosec}t-\sin t}$

$=\dfrac{\sec^2 t.\sin t}{\cos^2 t}=\sec^3 t.\tan t$

or $\left[\dfrac{d^2y}{dx^2}\right]_{t=\frac{\pi}{4}}=2\sqrt{2}\times 1=2\sqrt{2}$ ½

[CBSE Marking Scheme, 2019] (Modified)

15. $\dfrac{dx}{d\theta}=a(2-2\cos 2\theta)=4a\sin^2\theta$ ½

$\dfrac{dy}{d\theta}=2a\sin 2\theta=4a\sin\theta\cdot\cos\theta$ ½

$\therefore$ $\dfrac{dy}{dx}=\dfrac{4a\sin\theta\cos\theta}{4a\sin^2\theta}=\cot\theta$ **1**

$\left.\dfrac{dy}{dx}\right]_{\theta=\frac{\pi}{3}}=\dfrac{1}{\sqrt{3}}$ **1**

[CBSE Marking Scheme, 2018] (Modified)

Long Answer Type Questions

2. Given, $x \cos (a + y) = \cos y$

or $$x = \frac{\cos y}{\cos(a+y)}$$

On differentiating both sides *w.r.t.* y, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}\cos y - \cos y \frac{d}{dy}\cos(a+y)}{\cos^2(a+y)}$$ **1**

[By using quotient rule of derivative]

$$= \frac{\cos(a+y)\times(-\sin y)+\cos y\times\sin(a+y)}{\cos^2(a+y)}$$ ½

$$= \frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\cos^2(a+y)}$$

or $$\frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

[$\because \sin A \cos B - \cos A \sin B = \sin (A - B)$]

or $$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$ ...(i) ½

Again, on differentiating both sides of Eq. (i) *w.r.t.* x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{\sin a}\frac{d}{dx}\cos^2(a+y)$$

$$= \frac{1}{\sin a}\times\frac{d}{dy}\cos^2(a+y)\times\frac{dy}{dx}$$ **1**

$$= \frac{1}{\sin a}\times 2\cos(a+y)$$

$$[-\sin(a+y)]\times\frac{dy}{dx}$$

$$= -\frac{2\sin(a+y)\cos(a+y)}{\sin a}\times\frac{dy}{dx}$$ **1**

$$\therefore \frac{d^2y}{dx^2} = \frac{-\sin 2(a+y)}{\sin a}\frac{dy}{dx}$$ [$\because 2\sin\theta\cos\theta = \sin 2\theta$]

$$\therefore \sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$$ **1**

Hence Proved.

REFLECTIONS

- Will you be able to calculate the higher order derivatives?
- Do you think, derivatives are used in medical field?
- Can you apply derivatives to calculate profit and loss in business?

❑❑❑