CHAPTER

9

DIFFEREN-TIAL EQUATIONS

Syllabus

Definition, order and degree, general and particular solutions of a differential equation. Solution of differential equations by method of separation of variables, solutions of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type:

$\frac{dy}{dx} + py = q$, *where p and q are the functions of x or constants*

$\frac{dx}{dy} + px = q$, *where p and q are the functions of y or constants*

In this chapter you will study

- *The concept of differential equations using the idea of differential calculus.*
- *Identify order and degree of a differential equations when solution is not given.*
- *Various method to solve differential equations.*
- *How to solve linear differential equation and homogenous differential equation of first order and first degree.*

List of Topics

Topic-1 Basic Differential Equations

Concepts Covered • *Order of differential Equation* • *Degree of differential Equation*

Revision Notes

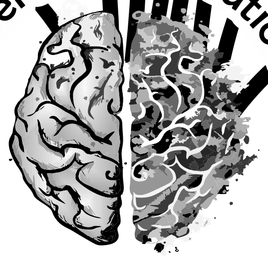
➢ **Orders and Degrees of Differential Equation :**

- We shall prefer to use the following notations for derivatives.
- $\frac{dy}{dx} = y', \frac{d^2y}{dx^2} = y'', \frac{d^3y}{dx^3} = y'''$
- For derivatives of higher order, it will be in convenient to use so many dashes as super suffix therefore, we use the notation y_n for n^{th} order derivative $\frac{d^ny}{dx^n}$.
- Order and degree (if defined) of a **differential equation** are always positive integers.

Key Words

Differential Equation: In Mathematics, a differential equation is an equation with one or more derivatives of a function. The derivative of the function is given by dy/dx. In other words, it is defined as the equation that contains derivatives of one or more dependent variables with respect to one or more independent variables.

Differential Equations



Definition

An equation involving derivatives of the dependent variable with respect to independent variable (variables) is called a differential equation. If there is only one independent variable, then we call it as an ordinary differential equation. *eg*: $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)=0$.

Order of a Differential Equation

It is the order of the highest order derivative occurring in the differential equation

eg: the order of $\frac{dy}{dx}=e^x$ is one and order of $\frac{d^2y}{dx^2}+x=0$ is two.

Degree of a Differential Equation

It is defined if the differential equation is a polynomial equation in its derivatives, and is defined as the highest power (positive integer only) of the highest order derivative.

eg: the degree of $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=0$ is three

Order and degree (if defined) of a differential equation are always positive integers.

Solution of a Differential Equation

A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.

eg: $y=e^x+1$ is a solution of $y''-y'=0$.

Since $y'=e^x$ and $y''=e^x \Rightarrow y''-y'=e^x-e^x=0$.

The order of a Differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

eg: Let the family of curves be

$y=mx,\ m=$ constant, then, $y'=m$

$y=y'x \Rightarrow y=\frac{dy}{dx}x \Rightarrow x\frac{dy}{dx}-y=0.$

Homogeneous Differential Equations

A differential equation which can be expressed in the form $\frac{dy}{dx}=f(x,y)$ or $\frac{dx}{dy}=g(x,y)$, where, $f(x,y)$ and $g(x,y)$ are homogeneous functions of degree zero is called a homogenous differential equation

eg: $(x^2+xy)dy=(x^2+y^2)dx$

To solve this, we substitute $y=vx$. and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Linear Differential Equations

The differential equation of the form $\frac{dy}{dx}+Py=Q$, where P, Q are constants or functions of 'x' only is called a first order linear differential equation. Its solution is given as $ye^{\int P.dx}=\int Q.e^{\int P.dx}dx+c$. *eg*: $\frac{dy}{dx}+3y=2x$ has solution $ye^{\int 3.dx}=\int 2x.e^{\int 3.dx}dx+c \Rightarrow ye^{3x}=2\int xe^{3x}dx+c$.

Variable Separation Method

It is used to solve such an equation in which variables can be separated completely.

eg: $ydx=xdy$ can be solved as $\frac{dx}{x}=\frac{dy}{y}$;

Integrating both sides

$\log x=\log y+\log c \Rightarrow \frac{x}{y}=c \Rightarrow x=cy$,

is the solution.

Trace the Mind Map

- First Level
- Second Level
- Third Level

Know the terms

- **Order of a differential equation:** It is the order of the highest order derivative appearing in the differential equation.
- **Degree of a differential equation:** It is the degree (power) of the highest order derivative, when the differential coefficients are made free from the radicals and the fractions.

OBJECTIVE TYPE QUESTIONS

Multiple Choice Questions

Q. 1. The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=x\sin\left(\frac{dy}{dx}\right) \text{ is}$$

(A) 1 **(B)** 2
(C) 3 **(D)** not defined

Ans. Option (D) is correct.

Explanation: The degree of above differential equation is not defined because when we expand $\sin\left(\frac{dy}{dx}\right)$ we get an infinite series in the increasing powers of $\frac{dy}{dx}$. Therefore, its degree is not defined.

Q. 2. The order and degree of the differential equation

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}+x^{1/5}=0 \text{ respectively, are}$$

(A) 2 and 4 **(B)** 2 and 2
(C) 2 and 3 **(D)** 3 and 3

Ans. Option (A) is correct.

Explanation:

Given that, $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}=-x^{1/5}$

$\Rightarrow \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}=-x^{1/5}$

$\Rightarrow \left(\frac{dy}{dx}\right)^{1/4}=-\left(x^{1/5}+\frac{d^2y}{dx^2}\right)$

On squaring both sides, we get

$$\left(\frac{dy}{dx}\right)^{1/2}=\left(x^{1/5}+\frac{d^2y}{dx^2}\right)^2$$

Again, on squaring both sides, we have

$$\frac{dy}{dx}=\left(x^{1/5}+\frac{d^2y}{dx^2}\right)^4$$

Order = 2, degree = 4

Q. 3. Which of the following is a second-order differential equation?

(A) $(y')^2+x=y^2$ **(B)** $y'y''+y=\sin x$
(C) $y'''+(y'')^2+y=0$ **(D)** $y'=y^2$

Ans. Option (B) is correct.

Explanation: The second-order differential equation is $y'y''+y=\sin x$.

Q. 4. The degree of differential equation

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+6y^5=0 \text{ is}$$

(A) 1 **(B)** 2
(C) 3 **(D)** 5

Ans. Option (A) is correct.

Explanation:

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+6y^5=0$$

We know that, the degree of a differential equation is exponent of highest order derivative.

$\therefore$ Degree = 1

Q. 5. The order of the differential equation

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0 \text{ is}$$

(A) 2 **(B)** 1
(C) 0 **(D)** not defined

Ans. Option (A) is correct.

Explanation :

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Find the order and the degree of the differential equation $x^2 \dfrac{d^2y}{dx^2} = \left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^4$.

R [CBSE Delhi Set-I, 2019]

Q. 2. Write the order and the degree of the following differential equation :

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

A I R [CBSE Delhi Set-III 2019]

Q. 3. Find the order and degree (if defined) of the differential equation.

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right)$$

R [CBSE OD Set-I, 2019]

Sol. Order = 2, Degree not defined ½+ ½

[CBSE Marking Scheme, 2019]

Detailed Solution :

Given differential equation is

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right)$$

The highest order derivative present in the differential equation is $\dfrac{d^2y}{dx^2}$. So, it is of order 2.

Clearly, the differential equation is not expressible as polynomial in $\dfrac{d^2y}{dx^2}$. So, its degree is not defined.

Hence, order = 2

degree = not defined

Commonly Made Error

- Mostly students write the degree as 1.

Answering Tip

- Note the questions in which degree is not defined.

Q. 4. Write the order of the differential equation:

$$\log\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dy}{dx}\right)^3 + x$$

A I R&U [SPQ 2018]

Q. 5. Write the sum of the order and degree of the differential equation $1+\left(\dfrac{dy}{dx}\right)^4 = 7\left(\dfrac{d^2y}{dx^2}\right)^3$.

A I R&U [Delhi Set I, II, III Comptt. 2015]

Sol. Degree of the given differential equation = 3

Order of the given differential equation = 2 ½

Hence, the sum of order and degree = 2 + 3

= 5 ½

[CBSE Marking Scheme 2015]

Short Answer Type Questions-I (2 marks each)

Q. 1. Show below is a differential equation.

$$y = e^{\sin\left(\frac{d^3y}{dx^3}\right)^2} + \left(\frac{dy}{dx}\right)^4$$

Find the order and the degree of the given differential equation. Give reasons to support your answer. [CBSE Practice Questions 2021-22]

Sol. Order is 3 as highest order derivative present in the given differential equation is $\dfrac{d^3y}{dx^3}$. **1**

Degree is not defined as the highest order derivative is the function of sine. **1**

Q. 2. Write the sum of the order and degree of the following differential equation:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 5$$

[SQP 2021-22]

Sol. Order = 2 **1**

Degree = 1 ½

Sum = 3 ½

[CBSE Marking Scheme, 2022]

Detailed Solution:

Given differential equation:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 5$$

$$\therefore \quad \frac{d^2y}{dx^2} = 5$$

Thus, degree = 2, order = 1 and sum = 2 + 1 = 3

Q. 3. Find the sum of the order and the degree of the following differential equations :

$$\frac{d^2y}{dx^2} + \sqrt[3]{\frac{dy}{dx}} + (1+x) = 0$$

A I R&U [SQP Dec. 2016-17]

Topic-2

Variable Separable Methods

Concepts Covered • *General Solution,* • *Particular Solutions ,* • *Variable Separable Method*

Revision Notes

➢ **Solutions of differential equations :**

(a) General Solution : The solution which contains as many as arbitrary constants as the order of the differential equations, *e.g.* $y = \alpha \cos x + \beta \sin x$ is the general solution of $\frac{d^2y}{dx^2} + y = 0$.

(b) Particular Solution : Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution *e.g.* $y = 3 \cos x + 2 \sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

(c) Solution of Differential by Variable Separable Method : A **variable** separable form of the differential equation is the one which can be expressed in the form of $f(x) dx = g(y)dy$. The solution is given by $\int f(x)dx = \int g(y)dy + k$, where k is the **constant** of integration.

Key Words

Variable: A value that keeps on changing is said to be variable. Variables are often represented by an alphabet like *a, b, c,* or *x, y, z*. Its value changes from time to time. ***e.g.:*** $3x + 5y = 7$ where x and y are variables that are changed according to the expression.

Constant: As the name implies, the constant is a value that remains constant ever. Constant has a fixed value and its value cannot be changed by any variable. Constants are represented by numbers.

e.g.: $3x + 5y = 7$, where 7 is the constant, we know its face value is 7 and it cannot be changed. But $3x$ and $5y$ are not constants because the variable x and y can change their value.

OBJECTIVE TYPE QUESTIONS

Multiple Choice Questions

Q. 1. The solution of differential equation $xdy - ydx = 0$ represents

(A) a rectangular hyperbola
(B) parabola whose vertex is at origin
(C) straight line passing through origin
(D) a circle whose centre is at origin

Ans. Option (C) is correct.

Explanation: Given that,

$$xdy - ydx = 0$$
$$\Rightarrow \quad xdy = ydx$$
$$\Rightarrow \quad \frac{dy}{y} = \frac{dx}{x}$$

On integrating both sides, we get

$$\log y = \log x + \log C$$
$$\Rightarrow \quad \log y = \log Cx$$
$$\Rightarrow \quad y = Cx$$

which is a straight line passing through the origin.

Q. 2. The general solution of $\frac{dy}{dx} = 2xe^{x^2-y}$ is

(A) $e^{x^2-y} = C$ **(B)** $e^{-y} + e^{x^2} = C$
(C) $e^{y} = e^{x^2} + C$ **(D)** $e^{x^2+y} = C$

Ans. Option (C) is correct.

Explanation: Given that,

$$\frac{dy}{dx} = 2xe^{x^2-y} = 2xe^{x^2} \cdot e^{-y}$$
$$\Rightarrow \quad e^y \frac{dy}{dx} = 2xe^{x^2}$$
$$\Rightarrow \quad e^y dy = 2xe^{x^2} dx$$

On integrating both sides, we get

$$\int e^y dy = 2\int xe^{x^2} dx$$

Put $x^2 = t$ in RHS integral, we get

$$2xdx = dt$$
$$\int e^y dy = \int e^t dt$$
$$\Rightarrow \quad e^y = e^t + C$$
$$\Rightarrow \quad e^y = e^{x^2} + C$$

Q. 3. The solution of equation $(2y-1)dx - (2x+3)dy = 0$ is

(A) $\frac{2x-1}{2y+3} = k$ (B) $\frac{2y+1}{2x-3} = k$

(C) $\frac{2x+3}{2y-1} = k$ (D) $\frac{2x-1}{2y-1} = k$

Ans. Option (C) is correct.

Explanation: Given that,

$$(2y-1)dx - (2x+3)dy = 0$$

$$\Rightarrow (2y-1)dx = (2x+3)dy$$

$$\Rightarrow \frac{dx}{2x+3} = \frac{dy}{2y-1}$$

On integrating both sides, we get

$$\frac{1}{2}\log(2x+3) = \frac{1}{2}\log(2y-1) + \log C$$

$$\Rightarrow \frac{1}{2}[\log\cdot(2x+3) - \log(2y-1)] = \log C$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{2x+3}{2y-1}\right) = \log C$$

$$\Rightarrow \left(\frac{2x+3}{2y-1}\right)^{1/2} = C$$

$$\Rightarrow \frac{2x+3}{2y-1} = C^2$$

$$\Rightarrow \frac{2x+3}{2y-1} = k,$$

where, $k = C^2$

Q. 4. The general solution of differential equation $(e^x+1)ydy = (y+1)e^x\,dx$ is

(A) $(y+1) = k(e^x+1)$

(B) $y+1 = e^x+1+k$

(C) $y = \log\{k(y+1)(e^x+1)\}$

(D) $y = \log \frac{x+1}{y+1} + k$

Ans. Option (C) is correct.

Explanation: Given differential equation

$$(e^x+1)ydy = (y+1)e^x dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x(1+y)}{(e^x+1)y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{(e^x+1)y}{e^x(1+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{e^x y}{e^x(1+y)} + \frac{y}{e^x(1+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{1+y} + \frac{y}{(1+y)e^x}$$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{1+y}\left(1+\frac{1}{e^x}\right)$$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{1+y}\left(\frac{e^x+1}{e^x}\right)$$

$$\Rightarrow \left(\frac{y}{1+y}\right)dy = \left(\frac{e^x}{e^x+1}\right)dx$$

On integrating both sides, we get

$$\int\frac{y}{1+y}dy = \int\frac{e^x}{1+e^x}dx$$

$$\Rightarrow \int\frac{1+y-1}{1+y}dy = \int\frac{e^x}{1+e^x}dx$$

$$\Rightarrow \int 1dy - \int\frac{1}{1+y}dy = \int\frac{e^x}{1+e^x}dx$$

$$\Rightarrow y - \log|(1+y)| = \log|(1+e^x)| + \log k$$

$$\Rightarrow y = \log(1+y) + \log(1+e^x) + \log(k)$$

$$\Rightarrow y = \log\left\{k(1+y)(1+e^x)\right\}$$

Q. 5. The numbers of arbitrary constants in the general solution of a differential equation of fourth order are:

(A) 0 (B) 2

(C) 3 (D) 4

Ans. Option (D) is correct.

Explanation: We know that the number of constants in the general solution of a differential equation of order n is equal to its order.

Therefore, the number of constants in the general Solution of fourth-order differential equation is four.

Q. 6. The numbers of arbitrary constants in the particular solution of a differential equation of second order are:

(A) 0 (B) 1

(C) 2 (D) 3 **[Board, 2020]**

Ans. Option (A) is correct.

Explanation: In the particular solution of a differential equation, there are no arbitrary constants.

OR

Topper Answer, 2020

A. 0 As it is a particular solution.

Q. 7. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

Ans. Option (A) is correct.

Explanation:

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$

$$\Rightarrow \quad \frac{dy}{e^y} = e^x dx$$

$$\Rightarrow \quad e^{-y}dy = e^x dx$$

Integrating both sides, we get:

$$\int e^{-y}dy = \int e^x dx$$

$$\Rightarrow \quad -e^{-y} = e^x + k$$

$$\Rightarrow \quad e^x + e^{-y} = -k$$

$$\Rightarrow \quad e^x + e^{-y} = C \qquad \text{(where, } C = -k\text{)}$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. How many arbitrary constants are there in the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$; $y(0) = 1$ R&U [CBSE SQP 2020-21]

Q. 2. Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$. A I

U [CBSE OD Set-II, 2019] [CBSE SQP-2020]

Sol. Given differential equation can be written as:

$$\frac{dy}{dx} = e^x.e^y \Rightarrow e^{-y}dy = e^x dx$$

Integrating both sides, we get

$$-e^{-y} = e^x + c$$ 1

[CBSE Marking Scheme, 2020]

Detailed Solution :

Given differential equation is

$$\frac{dy}{dx} = e^{x+y}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x e^y$$

$$\Rightarrow \quad \frac{dy}{e^y} = (e^x)dx$$

$$\Rightarrow \quad (e^{-y})dy = (e^x)dx$$

Integrating both sides, we get

$$\int (e^{-y})dy = \int (e^x)dx$$

$$\Rightarrow \quad -e^{-y} = e^x + c'$$

$$\Rightarrow \quad e^{-y} = -e^x + c \qquad [\text{where } c = -c']$$

Q. 3. Find the solution of the differential equation $\frac{dy}{dx} = x^3 e^{-2y}$.

A I R&U [O.D. Set I, II, III Comptt. 2015]

Q. 4. Write the solution of the differential equation $\frac{dy}{dx} = 2^{-y}$. R&U [Foreign 2015]

Sol. Given differential equation is

$$\frac{dy}{dx} = 2^{-y}$$

On separating the variables, we get

$$2^y\, dy = dx$$

On integrating both sides, we get

$$\int 2^y dy = \int dx$$

or $$\frac{2^y}{\log 2} = x + C_1$$

or $$2^y = x \log 2 + C_1 \log 2$$

$\therefore$ $$2^y = x \log 2 + C, \text{ where } C = C_1 \log 2$$ **1**

[CBSE Marking Scheme 2015]

Short Answer Type Questions-I (2 marks each)

AI **Q. 1. Solve the following differential equation: $\frac{dy}{dx} = x^3$ cosec y, given that $y(0) = 0$.**

R&U [CBSE SQP 2020-21]

Sol. $$\frac{dy}{dx} = x^3 \text{cosec}\, y \; ; \; y(0) = 0$$

$$\int \frac{dy}{\text{cosec}\, y} = \int x^3 dx$$ ½

$$\int \sin y\; dy = \int x^3 dx$$

$$-\cos y = \frac{x^4}{4} + c$$ 1

$$-1 = c \qquad (\because \; y = 0, \text{ when } x = 0)$$

$$\cos y = 1 - \frac{x^4}{4}$$ ½

[CBSE SQP Marking Scheme 2020]

Commonly Made Error

- Students forget to find the particular solution after finding the general solution.

Answering Tip

- Practice more problems based on finding particular solution.

Q. 2. Find the general solution of the differential equation.

$$xy\frac{dy}{dx} = (x+2)(y+2).$$ R&U [OD Comptt. 2017]

Sol. $\frac{y}{y+2}dy = \frac{(x+2)}{x}dx$ or

$$\int\left(1-\frac{2}{y+2}\right)dy = \int\left(1+\frac{2}{x}\right)dx \quad \mathbf{1}$$

$$y - 2\log|y+2| = x + 2\log|x| + C \quad \mathbf{1}$$

[CBSE Marking Scheme 2017]

Q. 3. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$; given that $y(0) = \sqrt{3}$.

R&U [OD Comptt. 2017]

Short Answer Type Questions-II (3 marks each)

Q. 1. Find the general solution of the differential equation given below.

$$\frac{dy}{dx} = \frac{1}{x(1+x^2)}$$

Show your steps.

[CBSE Practice Questions 2021-22]

Sol. Given differential equation is

$$\frac{dy}{dx} = \frac{1}{x(1+x^2)}$$

$$\Rightarrow \int dy = \int\frac{dx}{x(1+x^2)}$$

$$y = \int\frac{dx}{x(1+x^2)} \quad ...(i)$$

Now, $$\frac{1}{x(1+x^2)} = \frac{A}{x}+\frac{Bx+C}{1+x^2} \quad ...(ii)$$

[Using partial fraction]

$$1 = A(1+x^2) + (Bx+C)x$$

$$\Rightarrow 1 = (A+B)x^2 + Cx + A$$

$\therefore$ A + B = 0, C = 0 and A = 1

$\therefore$ A = 1, B = – 1 and C = 0

From eqn. (ii), we get

$$\frac{1}{x(1+x^2)} = \frac{1}{x}-\frac{x}{1+x^2}$$

Now, from eqn. (i), we get

$$y = \int\frac{1}{x}dx - \int\frac{x}{1+x^2}dx$$

or, $$y = \int\frac{1}{x}dx - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$

or, $$y = \log x - \frac{1}{2}\log(1+x^2) + \log c$$

or, $$y = \log x - \log\sqrt{(1+x^2)} + \log c$$

or, $$y = \log\left(\frac{cx}{\sqrt{1+x^2}}\right)$$

Q. 2. Find the general solution of the differential equation. $xdy = (e^y - 1)dx$

R&U [CBSE OD Set III-2020]

Sol. Given differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{x}(e^y - 1)$$

$$\Rightarrow \int\frac{dy}{e^y-1} = \int\frac{dx}{x} \quad ½$$

$$\Rightarrow \int\frac{e^{-y}}{1-e^{-y}}dy = \int\frac{dx}{x} \quad ½$$

$$\Rightarrow \log|1-e^{-y}| = \log|x| + \log C$$

$$\Rightarrow 1-e^{-y} = Cx \quad \mathbf{2}$$

[CBSE Marking Scheme 2020] (Modified)

Detailed Solution:

$$\frac{dy}{dx}+\frac{1}{x} = \frac{e^y}{x}$$

$$\frac{dy}{dx} = \frac{e^y-1}{x}$$

$$\frac{1}{e^y-1}dy = \frac{1}{x}dx$$

Integrating both side

$$\int\frac{1}{e^y-1}dy = \int\frac{1}{x}dx$$

$$\int\frac{1}{e^y(1-e^{-y})}dy = \int\frac{1}{x}dx$$

$$\int\frac{e^{-y}}{1-e^{-y}}dy = \int\frac{1}{x}dx$$

Let $1-e^{-y} = t$

$$0 - e^{-y}(-1)\,dy = dt$$

$$e^{-y}\,dy = dt$$

$$\int\frac{1}{t}dt = \int\frac{1}{x}dx$$

$$\log t = \log x + \log C$$

$$\log(1-e^{-y}) = \log xC$$

$$1-e^{-y} = xC$$

$$1 - xC = e^{-y}$$

$$1 - xC = \frac{1}{e^y}$$

$$e^y = \frac{1}{1-xC}$$

$$y = -\log(1-xC)$$

Commonly Made Error

- Students could not recognize the form of differential equation correctly.

Answering Tip

- Learn how to distinguish between variable separable, homogeneous and linear differential equations.

Q. 3. Find the particular solution of the following differential equation :

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1;\ y = 0 \text{ when } x = 0.$$

A I R&U [NCERT] [CBSE Delhi Set-III, 2019]
[CBSE OD Set I,II,III-2020]

Q. 4. Find the particular solution of the following differential equation.

$$\cos y\, dx + (1 + 2e^{-x}) \sin y\, dy = 0;\ y(0) = \frac{\pi}{4}$$

R&U [SQP 2018-19]

Sol. $\cos y\, dx + (1 + 2e^{-x}) \sin y\, dy = 0$

$\Rightarrow \quad \int \frac{dx}{1+2e^{-x}} = \int \frac{-\sin y}{\cos y} dy$ ½

$\Rightarrow \quad \int \frac{e^x}{2+e^x} dx = \int \frac{-\sin y}{\cos y} dy$

$\Rightarrow \quad \text{In}(e^x + 2) = \text{In}\ |\cos y| + \text{In C}$

$\Rightarrow \quad \text{In}(e^x + 2) = \text{In}\ |\cos y|\ \text{C}$

$\Rightarrow \quad e^x + 2 = \text{C}\ |\cos x|$ 1

$\Rightarrow \quad e^x + 2 = \pm \text{C} \cos y \Rightarrow e^x + 2 = k \cos y$ ...(1)

Substituting $x = 0, y = \frac{\pi}{4}$ in (1), we get

$1 + 2 = k \cos \frac{\pi}{4}$

$\Rightarrow \quad k = 3\sqrt{2}$ 1

$\therefore \quad e^x + 2 = 3\sqrt{2} \cos y$ is the particular solution. ½

[CBSE Marking Scheme 2018-19] (Modified)

Q. 5. Find the particular solution of the following differential equation :

$$xy\frac{dy}{dx} = (x + 2)(y + 2);\ y = -1, \text{ when } x = 1.$$

A I R&U [OD Comptt. 2017]

Sol. $xy\frac{dy}{dx} = (x + 2)(y + 2)$

or $\frac{y}{y+2} dy = \frac{x+2}{x} dx$

or $\left(1 - \frac{2}{y+2}\right) dy = \left(1 + \frac{2}{x}\right) dx$ 1

On integrating it, we get

$y - 2\log(y + 2) = x + 2\log x + C$ ...(i) 1

Given $y = -1$, when $x = 1$, then from (i)

$-1 - 2\log(-1 + 2) = 1 + 2\log 1 + C$

or $C = -2$, as $\log 1 = 0$ ½

Then (i) becomes :

$y - 2\log(y + 2) = x + 2\log x - 2$

or $y = x - 2 + 2\{\log(y+2) + \log x\}$

or $y = x - 2 + 2\log\{x(y + 2)\}$

This is the required particular solution. ½

Long Answer Type Questions (5 marks each)

AI **Q. 1. Solve the following differential equation.**

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

R&U [Foreign 2015]

Sol. Given differential equation is

$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$

or $\sqrt{(1+x^2)+y^2(1+x^2)} = -xy\frac{dy}{dx}$

or $\sqrt{(1+x^2)(1+y^2)} = -xy\frac{dy}{dx}$

or $\sqrt{1+x^2}.\sqrt{1+y^2} = -xy\frac{dy}{dx}$

or $\frac{y}{\sqrt{1+y^2}} dy = -\frac{\sqrt{1+x^2}}{x} dx$ 1

On integrating both sides, we get

$\int \frac{y}{\sqrt{1+y^2}} dy = -\int \frac{\sqrt{1+x^2}}{x^2}.x dx$ 1

On putting $1 + y^2 = t$ and $1 + x^2 = u^2$

or $2y\,dy = dt$ and $2x\,dx = 2u\,du$

$\Rightarrow \quad y\,dy = \frac{dt}{2}$

and $x\,dx = u\,du$ 1

$\therefore \quad \frac{1}{2}\int t^{-1/2} dt = -\int \frac{u}{u^2-1}.u\,du$

or $\frac{1}{2}\int t^{-1/2} dt = -\int \frac{u^2}{u^2-1} du$

or $\frac{1}{2}\frac{t^{1/2}}{\frac{1}{2}} = -\int \frac{(u^2-1+1)}{u^2-1} du$ 1

or $t^{1/2} = -\int \frac{u^2-1}{u^2-1} du - \int \frac{1}{u^2-1} du$

or $\sqrt{1+y^2} = -\int du - \int \frac{1}{u^2-(1)^2} du$

[put $1 + y^2 = t$]

or $\sqrt{1+y^2} = -u - \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + C$ **1**

$\left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

$\therefore \sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C$

which is the required solution. **1**

[CBSE Marking Scheme 2015] (Modified)

Topic-3 Linear Differential Equations

Concepts Covered • *Linear Differential Equations in x only and in y only*

Revision Notes

- **Solutions of Differential Equations:**
  - **Linear differential equation in y :** It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are functions of x only.
- **Solving Linear Differential Equation in y :**

STEP 1 : Write the given differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

STEP 2 : Find the **Integration Factor** (*I.F.*) = $e^{\int P(x)dx}$.

Key Word

Integrating Factor: An integrating factor is a function by which an ordinary differential equation can be multiplied in order to make it integrable.

STEP 3 : The solution is given by, $y.(I.F.) = \int Q(x).(I.F.)dx + k$, where k is the constant of integration.

- **Linear differential equation in x :** It is of the form $\frac{dx}{dy} + P(y)x = Q(y)$, where $P(y)$ and $Q(y)$ are functions of y only.
- **Solving Linear Differential Equation in x :**

STEP 1 : Write the given differential equation in the form $\frac{dx}{dy} + P(y)x = Q(y)$.

STEP 2 : Find the Integration Factor (*I.F.*) = $e^{\int P(y)dy}$.

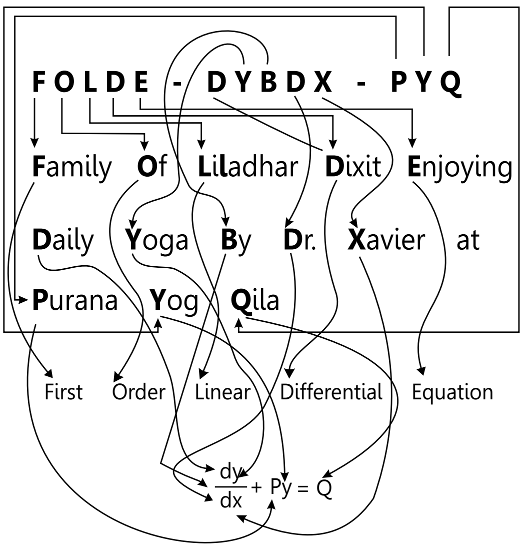
STEP 3 : The solution is given by, $x.(I.F.) = \int Q(y).(I.F.)dy + \lambda$, where λ is the **constant of integration**.

Key Word

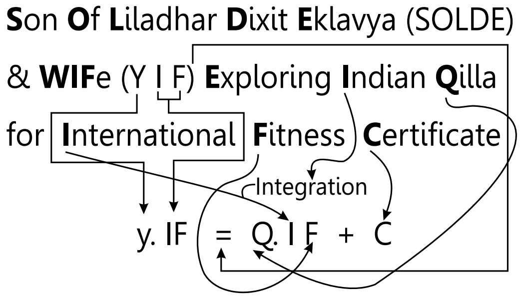
Constant of integration: A constant that is added to the function obtained by evaluating the indefinite integral of a given function, indicating that all indefinite integrals of the given function differ by, at most, a constant.

Mnemonics

Linear Differential Equations



SOLDE-YIF-EIQ-IFC



S — Solution
O — Of
L — Linear
D — Differential
E — Equation

Interpretation :

Differential equation is of the form $\frac{dy}{dx}+py=Q$, where P and Q are constants or the function of 'x' is called a first order linear differential equations. Its solution is given as

$$Y.IF=\int Q.IF+C$$

OBJECTIVE TYPE QUESTIONS

Multiple Choice Questions

Q. 1. The integrating factor of differential equation $\cos x\frac{dy}{dx}+y\sin x=1$ is

(A) $\cos x$ **(B)** $\tan x$

(C) $\sec x$ **(D)** $\sin x$

Ans. Option (C) is correct.

Explanation: Given that,

$$\cos x\frac{dy}{dx}+y\sin x = 1$$

$$\Rightarrow \quad \frac{dy}{dx}+y\tan x = \sec x$$

Here, $P = \tan x$ and $Q = \sec x$

$$\text{IF} = e^{\int Pdx}$$
$$= e^{\int \tan x dx}$$
$$= e^{\ln \sec x}$$
$$\therefore \quad \text{IF} = \sec x$$

Q. 2. The integrating factor of differential equation $(1-x^2)\frac{dy}{dx}-xy=1$ is

(A) $-x$ **(B)** $\frac{x}{1+x^2}$

(C) $\sqrt{1-x^2}$ **(D)** $\frac{1}{2}\log(1-x^2)$

Ans. Option (C) is correct.

Explanation: Given that,

$$(1-x^2)\frac{dy}{dx}-xy = 1$$

$$\Rightarrow \quad \frac{dy}{dx}-\frac{x}{1-x^2}y = \frac{1}{1-x^2}$$

which is a linear differential equation.

$$\text{IF} = e^{-\int\frac{x}{1-x^2}dx}$$

Put $\quad 1-x^2 = t$

$$\Rightarrow \quad -2xdx = dt$$

$$\Rightarrow \quad xdx = -\frac{dt}{2}$$

Now, $\quad \text{IF} = e^{\frac{1}{2}\int\frac{dt}{t}}$

$$= e^{\frac{1}{2}\log t}$$
$$= e^{\frac{1}{2}\log(1-x^2)}$$
$$= \sqrt{1-x^2}$$

Q. 3. The solution of $x\frac{dy}{dx}+y=e^x$ is

(A) $y=\frac{e^x}{x}+\frac{k}{x}$ **(B)** $y = xe^x + Cx$

(C) $y = xe^x + k$ **(D)** $x=\frac{e^y}{y}+\frac{k}{y}$

Ans. Option (A) is correct.

Explanation: Given that,

$$x\frac{dy}{dx}+y = e^x$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x} = \frac{e^x}{x}$$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int\frac{1}{x}dx}$$
$$= e^{(\log x)}$$
$$= x$$

The general solution is

$$y\cdot x = \int\left(\frac{e^x}{x}\cdot x\right)dx$$

$$\Rightarrow \quad y\cdot x = \int e^x dx$$

$$\Rightarrow \quad y \cdot x = e^x + k$$

$$\Rightarrow \quad y = \frac{e^x}{x} + \frac{k}{x}$$

Q. 4. The solution of

$\frac{dy}{dx} + y = e^{-x}$, $y(0) = 0$ **is**

(A) $y = e^{-x}(x-1)$ **(B)** $y = xe^x$

(C) $y = xe^{-x} + 1$ **(D)** $y = xe^{-x}$

Ans. Option (D) is correct.

Explanation: Given that,

$$\frac{dy}{dx} + y = e^{-x}$$

which is a linear differential equation.

Here, $P = 1$ and $Q = e^{-x}$

$$\text{IF} = e^{\int dx} = e^x$$

The general solution is

$$y \cdot e^x = \int e^{-x} \cdot e^x dx + C$$

$$\Rightarrow \quad ye^x = \int dx + C$$

$$\Rightarrow \quad ye^x = x + C \quad \text{...(i)}$$

When $x = 0$ and $y = 0$ then, $0 = 0 + C \Rightarrow C = 0$

eqn. (i) becomes $y \cdot e^x = x \Rightarrow y = xe^{-x}$

Q. 5. The general solution of $\frac{dy}{dx} + y\tan x = \sec x$ **is**

(A) $y\sec x = \tan x + C$

(B) $y\tan x = \sec x + C$

(C) $\tan x = y\tan x + C$

(D) $x\sec x = \tan y + C$

Ans. Option (A) is correct.

Explanation: Given differential equation is

$$\frac{dy}{dx} + y\tan x = \sec x$$

which is a linear differential equation

Here, $P = \tan x$, $Q = \sec x$,

$$\therefore \quad \text{IF} = e^{\int \tan x dx} = e^{\log|\sec x|} = \sec x$$

The general solution is

$$y \cdot \sec x = \int \sec x \cdot \sec x \, dx + C$$

$$\Rightarrow \quad y \cdot \sec x = \int \sec^2 x \, dx + C$$

$$\Rightarrow \quad y \cdot \sec x = \tan x + C$$

Q. 6. The solution of differential equation

$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}$ **is**

(A) $y(1+x^2) = C + \tan^{-1} x$

(B) $\frac{y}{1+x^2} = C + \tan^{-1} x$

(C) $y\log(1+x^2) = C + \tan^{-1} x$

(D) $y(1+x^2) = C + \sin^{-1} x$

Ans. Option (A) is correct.

Explanation: Given that,

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}$$

Here, $P = \frac{2x}{1+x^2}$

and $Q = \frac{1}{(1+x^2)^2}$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{2x}{1+x^2} dx}$$

Put $1 + x^2 = t$

$\Rightarrow \quad 2xdx = dt$

$$\therefore \quad \text{IF} = e^{\int \frac{dt}{t}} = e^{\log t} = e^{\log(1+x^2)} = 1 + x^2$$

The general solution is

$$y \cdot (1+x^2) = \int (1+x^2) \frac{1}{(1+x^2)^2} + C$$

$$\Rightarrow \quad y(1+x^2) = \int \frac{1}{1+x^2} dx + C$$

$$\Rightarrow \quad y(1+x^2) = \tan^{-1} x + C$$

Q. 7. The Integrating Factor of the differential equation

$x\frac{dy}{dx} - y = 2x^2$ **is**

(A) e^{-x} **(B)** e^{-y}

(C) $\frac{1}{x}$ **(D)** x

Ans. Option (C) is correct.

Explanation: The given differential equation is:

$$x\frac{dy}{dx} - y = 2x^2$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 2x$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + Py = Q$$

(where $P = -\frac{1}{x}$ and $Q = 2x$)

The integrating factor (IF) is given by the relation,

$$\therefore \quad \text{IF} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log(x^{-1})} = x^{-1} = \frac{1}{x}$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Write the integrating factor of the differential equations $\sqrt{x}\frac{dy}{dx}+y=e^{-2\sqrt{x}}$.

R&U [O. D. Set I, II, III Comptt. 2015]

Q. 2. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$.

R&U [NCERT][Delhi 2015]

Sol. We know, $I.F. = e^{\int p\,dx}$

Here, $P = \frac{1}{\sqrt{x}}$ ½

$\therefore$ $I.F. = e^{\int \frac{1}{\sqrt{x}}dx}$

$= e^{2\sqrt{x}}$ ½

[CBSE Marking Scheme 2015]

Q. 3. Write the integrating factor of the following differential equation.

$$(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0$$

R&U [All India 2015]

Sol. Given differential equation is

$$(1+y^2)+(2xy-\cot y)\frac{dy}{dx} = 0.$$

The above equation can be rewritten as

$$(\cot y - 2xy)\frac{dy}{dx} = 1 + y^2$$

or $$\frac{\cot y - 2xy}{(1+y^2)} = \frac{dx}{dy}$$

or $$\frac{dx}{dy} = \frac{\cot y}{1+y^2}-\frac{2xy}{1+y^2}$$

or $$\frac{dx}{dy}+\frac{2y}{1+y^2}.x = \frac{\cot y}{1+y^2}$$ ½

which is a linear differential equation of the form $\frac{dx}{dy}+Px = Q$, where $P = \frac{2y}{1+y^2}$ and $Q = \frac{\cot y}{1+y^2}$.

Now, integrating factor $= e^{\int P\,dy} = e^{\int \frac{2y}{1+y^2}dy}$

Put $1 + y^2 = t$ or $2y\,dy = dt$

$\therefore$ $IF = e^{\int \frac{dt}{t}} = e^{\log|t|} = t = 1+y^2$ ½

[CBSE Marking Scheme 2015]

Short Answer Type Questions-I (2 marks each)

Q. 1. Find the general solution of the differential equation $\frac{dy}{dx}+2y=e^{3x}$ R&U [Delhi Comptt. 2017]

Sol. Integrating factor is $e^{\int 2dx} = e^{2x}$ ½

$\therefore$ Required solution is

$y.e^{2x} = \int e^{3x}.e^{2x}dx$ ½

$y.e^{2x} = \frac{e^{5x}}{5}+C$ ½

or $y = \frac{e^{3x}}{5}+Ce^{-2x}$ ½

[CBSE Marking Scheme 2017]

Q. 2. Find the general solution of the differential equation $\frac{dy}{dx}+\frac{2}{x}y = x$ R&U [Delhi Comptt. 2017]

Sol. Integrating factor is

$e^{\int \frac{2}{x}dx} = x^2$ ½

Solution is $y.x^2 = \int x.x^2dx+C$ ½

$y.x^2 = \frac{x^4}{4} + C$

or $y = \frac{x^2}{4}+\frac{C}{x^2}$ 1

[CBSE Marking Scheme 2017]

Commonly Made Error

- Many candidates do not express the answer in terms of constant 'C'.

Answering Tip

- Give adequate practice on various type.

Q. 3. Find the integrating factor of the differential equation $\frac{dy}{dx}+y = \frac{1+y}{x}$

R&U [OD Comptt. 2017]

Short Answer Type Questions-II (3 marks each)

Q. 1. Find the particular solution of the following differential equation, given that $y = 0$ when $x = \frac{\pi}{4}$: $\frac{dy}{dx} + y\cot x = \frac{2}{1+\sin x}$ [SQP 2021-2022]

Sol. The differential equation a linear differential equation

$$\text{IF} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x \quad 1$$

The general solution is given by

$$y\sin x = \int 2\frac{\sin x}{1+\sin x}dx$$

$$\Rightarrow \quad y\sin x = 2\int \frac{\sin x + 1 - 1}{1+\sin x}dx$$

$$\Rightarrow \quad y\sin x = 2\int \left[1 - \frac{1}{1+\sin x}\right]dx \quad ½$$

$$\Rightarrow \quad y\sin x = 2\int \left[1 - \frac{1}{1+\cos\left(\frac{\pi}{2}-x\right)}\right]dx$$

$$\Rightarrow \quad y\sin x = 2\int \left[1 - \frac{1}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]dx$$

$$\Rightarrow \quad y\sin x = 2\int \left[1 - \frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]dx$$

$$\Rightarrow \quad y\sin x = 2\int \left[x + \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right] + c \quad 1$$

Given that $y = 0$, when $x = \frac{\pi}{4}$

Hence, $0 = 2\left[\frac{\pi}{4} + \tan\frac{\pi}{8}\right] + c$

Hence, the particular solution is

$$y = \text{cosec}x\left[2\left\{x + \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} - \left(\frac{\pi}{2} + 2\tan\frac{\pi}{8}\right)\right] \quad ½$$

[CBSE Marking Scheme 2022]

Q. 2. Find the general solution of the following differential equation: $x\,dy - (y + 2x^2)dx = 0$

A I R&U [CBSE SQP 2020-21]

Sol. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y+2x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = 2x$$

Here $\quad P = -\frac{1}{x}$

$Q = 2x$ ½

$$\text{IF} = e^{\int P dx}$$

$$= e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x} \quad 1$$

The solutions is :

$$y \times \frac{1}{x} = \int\left(2x \times \frac{1}{x}\right)dx \quad 1$$

$$\Rightarrow \quad \frac{y}{x} = 2x + c$$

$$\Rightarrow \quad y = 2x^2 + cx \quad ½$$

[CBSE SQP Marking Scheme 2020-21]

Commonly Made Error

- Some students think that it is homogeneous and put $\frac{y}{x} = v$ and goes wrong.

Answering Tip

- In homogeneous equation, the degree of all terms will be the same.

Q. 3. Solve $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ subject to the initial condition $y(0) = 0$.

R&U [CBSE Delhi Set I-2019]

[Delhi Set I, II, III Comptt. 2016]

Sol. Given differential equation can be written as :

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{4x^2}{1+x^2},$$

with $\quad P = \frac{2x}{1+x^2},\ Q = \frac{4x^2}{1+x^2}$ ½

I.F. (Integrating factor)

$$= e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\log(1+x^2)} = 1 + x^2 \quad ½$$

∴ General solution is :

$$y(1+x^2) = \int \frac{4x^2}{1+x^2}\cdot(1+x^2)dx + C \quad 1$$

or $\quad y\cdot(1+x^2) = \frac{4x^3}{3} + C$ ½

Putting $x = 0$ and $y = 0$, we get $C = 0$

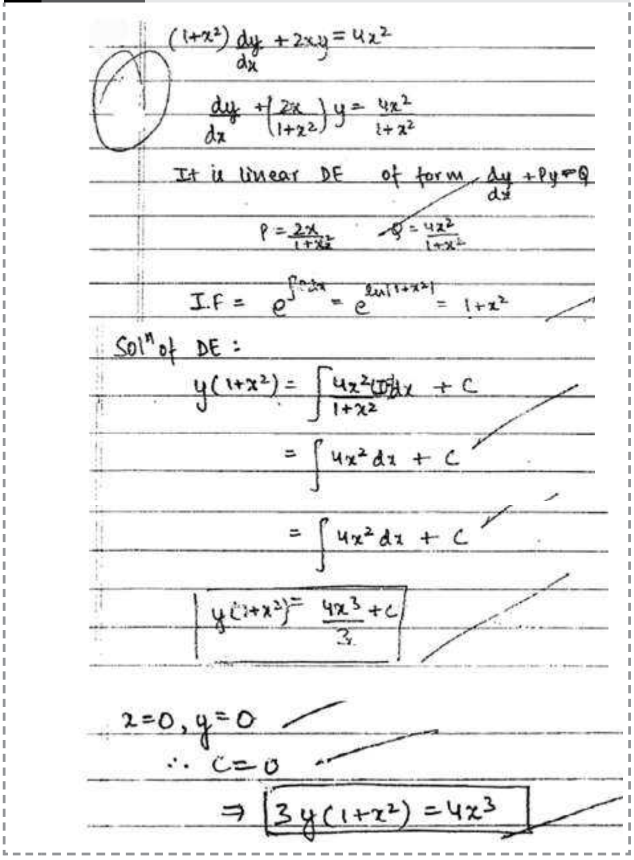
∴ Solution is :

$$y = \frac{4x^3}{3(1+x^2)} \quad ½$$

[CBSE Marking Scheme 2019] (Modified)

OR

Topper Answer, 2020


$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$

It is linear DE of form $\frac{dy}{dx} + Py = Q$

$P = \frac{2x}{1+x^2}$, $Q = \frac{4x^2}{1+x^2}$

$I.F = e^{\int \frac{2x}{1+x^2}} = e^{\ln|1+x^2|} = 1+x^2$

Sol^n of DE :

$y(1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)dx + C$

$= \int 4x^2 dx + C$

$= \int 4x^2 dx + C$

$y(1+x^2) = \frac{4x^3}{3} + c$

$x = 0, y = 0$

$\therefore C = 0$

$\Rightarrow 3y(1+x^2) = 4x^3$

Q. 4. Solve the differential equation : $\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2 + 2$

R&U [CBSE Delhi Set-III, 2019]

Sol. I.F. $= e^{-\int \frac{2x}{1+x^2}dx} = \frac{1}{1+x^2}$ ½

Solution is given by,

$$y \cdot \left(\frac{1}{1+x^2}\right) = \int \frac{x^2+2}{1+x^2}dx$$ ½

$$y \cdot \frac{1}{1+x^2} = \int\left(1 + \frac{1}{1+x^2}\right)dx = x + \tan^{-1}x + c$$ 1

or $y = (1+x^2)(x + \tan^{-1}x + c)$ 1

[CBSE Marking Scheme, 2019] (Modified)

Detailed Solution :

Given differential equation is :

$$\frac{dy}{dx} - \left(\frac{2x}{1+x^2}\right)y = x^2 + 2$$

On comparing the given differential equation with

$$\frac{dy}{dx} + Py = Q, \text{ we get}$$

$$P = -\frac{2x}{1+x^2}, Q = x^2 + 2$$

$$\therefore \quad \text{I.F.} = e^{\int P dx}$$

$$= e^{\int \frac{-2x}{1+x^2}dx} = e^{-\log(1+x^2)}$$

$$= \frac{1}{1+x^2}$$

Solution is given by :

$$y\,(I.F) = \int Q \times I.F\,dx$$

$$\Rightarrow \quad y\frac{1}{1+x^2} = \int \frac{x^2+2}{1+x^2}dx = \int \frac{1+x^2+1}{1+x^2}dx$$

$$\Rightarrow \quad \frac{y}{1+x^2} = \int 1\,dx + \int \frac{1}{1+x^2}dx$$

$$\Rightarrow \quad \frac{y}{1+x^2} = x + \tan^{-1}x + c$$

$$\Rightarrow \quad y = (1+x^2)(x + \tan^{-1}x + c)$$

Q. 5. Find the general solution of the differential equation:

$$\frac{dx}{dy} = \frac{y\tan y - x\tan y - xy}{y\tan y}$$

R&U [SPQ 2018-19]

Sol. Given, $\frac{dx}{dy} = \frac{y\tan y - x\tan y - xy}{y\tan y}$

$$\frac{dx}{dy} + \left(\frac{1}{y} + \frac{1}{\tan y}\right)x = 1$$ 1

$$I.F. = e^{\int\left(\frac{1}{y} + \cot y\right)dy} = e^{\ln y + \ln \sin y}$$

$$I.F. = e^{\ln(y \sin y)} = y \sin y$$ ½

Solution of the D.E. is:

$$x \times I.F. = \int (Q \times I.F.)dy$$

$$\Rightarrow \quad xy \sin y = \int y \sin y\,dy$$ ½

$$\Rightarrow \quad xy \sin y = y(-\cos y) - \int(-\cos y)dy$$

$$\Rightarrow \quad xy \sin y = -y\cos y + \sin y + C$$

$$\Rightarrow \quad x = \frac{\sin y - y\cos y + C}{y \sin y}$$ 1

[CBSE Marking Scheme 2018,] (Modified)

Q. 6. Find the particular solution of the differential equation $\frac{dy}{dx} + 2y\tan x = \sin x$, given that $y = 0$ when $x = \frac{\pi}{3}$. A I R&U [Foreign, 2014] [CBSE Delhi/OD-2018] [NCERT]

Q. 7. Solve the differential equation $x\frac{dy}{dx} + y = x\cos x + \sin x$, given $y\left(\frac{\pi}{2}\right) = 1$.

A I R&U [Delhi 2017]

Sol. $x\dfrac{dy}{dx} + y = x\cos x + \sin x$

or $\dfrac{dy}{dx} + \dfrac{1}{x}\cdot y = \cos x + \dfrac{1}{x}\sin x$ ½

$\therefore$ $I.F. = e^{\int \frac{1}{x}dx} = e^{\log x} = x$ ½

$\therefore$ Solution is $xy = \int (x\cos x + \sin x)dx$ ½

or $xy = x\sin x + C$ **1**

$\therefore$ $y = \sin x + C\dfrac{1}{x}$

$x = \dfrac{\pi}{2}, y = 1$ or $1 = 1 + C\left(\dfrac{2}{\pi}\right)$ or $C = 0$

$\therefore$ Solution is $y = \sin x$. ½

[CBSE Marking Scheme 2017] (Modified)

Q. 8. Find the particular solution of the differential equation :

$$\frac{dy}{dx} + y\cot x = 2x + x^2\cot x, x \neq 0.$$

Given that $y = 0$, when $x = \dfrac{\pi}{2}$.

A I R&U [Delhi 2017] [NCERT]

Sol. The given equation is a linear differential equation of the type $\dfrac{dy}{dx} + Py = Q$,

where $P = \cot x, Q = 2x + x^2\cot x$

$\therefore$ $I.F. = e^{\int \cot x\,dx} = e^{\log\sin x}$

$= \sin x$ ½

Hence, the solution of the differential equation is given by :

$$y.\sin x = \int (2x + x^2\cot x).\sin x\,dx + C$$

$$= \int 2x.\sin x\,dx + \int x^2\cos x\,dx + C$$

$$= \sin x.\frac{2x^2}{2} - \int \cos x\frac{2x^2}{2}dx + \int x^2\cos x\,dx + C$$

$= x^2\sin x + C$ ...(i) **1**

Substituting $y = 0$ and $x = \dfrac{\pi}{2}$ in the above equation (i), we get

$$0 = \frac{\pi^2}{4}\times 1 + C$$

or $C = -\dfrac{\pi^2}{4}$ ½

Now substituting the value of C in eq. (i), we get

$y\sin x = x^2\sin x - \dfrac{\pi^2}{4}$ **1**

or $y = x^2 - \dfrac{\pi^2}{4\sin x}$ $(\sin x \neq 0)$

This is the required particular solution of the given differential equation.

[CBSE Marking Scheme 2017] (Modified)

Q. 9. Solve the differential equation $(\tan^{-1} x - y)dx = (1 + x^2)\,dy$

R&U [O.D. Set I 2017]

Long Answer Type Questions (5 marks each)

Q. 1. Solve the differential equation :

$\dfrac{dy}{dx} - 3y\cot x = \sin 2x$ given $y = 2$, when $x = \dfrac{\pi}{2}$.

A I R&U [NCERT]

[O.D. Set I, II, III Comptt. 2015] [Foreign, 2017]

Sol. $\dfrac{dy}{dx} - 3\cot x\cdot y = \sin 2x$ **1**

$I.F. = e^{\int -3\cot x\,dx}$

$= e^{-3\log(\sin x)} = (\sin x)^{-3}$

$= \text{cosec}^3 x$ **1**

$\therefore$ Solution is

$$y\cdot\text{cosec}^3 x = \int \sin 2x\cdot\text{cosec}^3 x\,dx$$

$= \int 2\text{cosec}\,x\cot x\,dx$ **1**

or $y\cdot\text{cosec}^3 x = -2\,\text{cosec}\,x + C$

or $y = -2\sin^2 x + C\sin^3 x$ **1**

At $x = \dfrac{\pi}{2}, y = 2$

or $C = 4$

$\therefore$ $y = -2\sin^2 x + 4\sin^3 x$ **1**

[CBSE Marking Scheme 2017]

Q. 2. Solve the differential equation $x\dfrac{dy}{dx} + y = x\cos x + \sin x$, given that $y = 1$ when $x = \dfrac{\pi}{2}$

R&U [Delhi, 2017]

Q. 3. Find the particular solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1}y})\dfrac{dy}{dx} = 0$, given that $y = 0$ when $x = 1$.

R&U [Foreign 2017]

Sol. Given differential equation can be written as

$\dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{e^{\tan^{-1}y}}{1+y^2}$ **1**

$\text{I.F.} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$ ½

Solution is given by

$xe^{\tan^{-1}y} = \int \dfrac{e^{\tan^{-1}y}}{1+y^2}\times e^{\tan^{-1}y}dy = \int \dfrac{e^{2\tan^{-1}y}}{1+y^2}dy$ **1** or $\quad xe^{\tan^{-1}y} = \dfrac{e^{2\tan^{-1}y}}{2} + c$ **1**

when $\quad x = 1, y = 0$ or $c = \dfrac{1}{2}$ **1**

$\therefore$ Solution is given by $xe^{\tan^{-1}y} = \dfrac{1}{2}e^{2\tan^{-1}y} + \dfrac{1}{2}$

or $x = \dfrac{1}{2}(e^{\tan^{-1}y} + e^{-\tan^{-1}y})$ ½

[CBSE Marking Scheme, 2017] (Modified)

Q. 4. Find the particular solution of the differential equation $(\tan^{-1}y - x)\,dy = (1 + y^2)\,dx$, given that when $x = 0, y = 0$. R&U [OD 2015]

Sol. Given $(\tan^{-1}y - x)\,dy = (1 + y^2)\,dx$

or $\quad \dfrac{dx}{dy} = \dfrac{\tan^{-1}y - x}{1+y^2}$ ½

or $\quad \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1}y}{1+y^2}$ ...(i) ½

This is a linear differential equation with :

$P = \dfrac{1}{1+y^2}$ and $Q = \dfrac{\tan^{-1}y}{1+y^2}$ ½

or $\quad I.F. = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$ ½

Multiplying both sides of eqn. (i) by

$I.F = e^{\tan^{-1}y}$, we get

$x.I.F. = \int Q.I.F.dy$ **1**

$xe^{\tan^{-1}y} = \int \dfrac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}\,dy + c$ ½

or $\quad xe^{\tan^{-1}y} = \int te^t dt + C,$ where $t = \tan^{-1}y$

or $\quad xe^{\tan^{-1}y} = e^t(t-1) + C$ ½

or $\quad xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$ ...(ii)

It is given that $y(0) = 0$ *i.e.*, $y = 0$ when $x = 0$

Putting $x = 0, y = 0$ in eqn. (ii), we get

$0 = e^0(0-1) + c$ or $c = 1$ ½

Putting $c = 1$ in eq. (ii), we get

$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + 1$

or $\quad (x - \tan^{-1}y + 1)\,e^{\tan^{-1}y} = 1$ ½

[CBSE Marking Scheme 2015] (Modified)

Q. 5. Find the particular solutions of differential equation: $\dfrac{dy}{dx} = \dfrac{x + y\cos x}{1+\sin x}$ given that $y = 1$ when $x = 0$.

Sol. Since, $\quad \dfrac{dy}{dx} = \dfrac{-x}{1+\sin x} - \dfrac{y\cos x}{1+\sin x}$

or $\quad \dfrac{dy}{dx} + \dfrac{y\cos x}{1+\sin x} = \dfrac{-x}{1+\sin x}$, ...(i) ½

which is a linear differential equation with

$P = \dfrac{\cos x}{1+\sin x}$; $Q = \dfrac{-x}{1+\sin x}$ **1**

$\therefore$ Integrating factor

$\text{I.F.} = e^{\int \frac{\cos x}{1+\sin x}dx} = e^{\log(1+\sin x)}$

$= 1 + \sin x$ **1**

For general solution, we have

$y(1 + \sin x) = \int -x\,dx + C$

$\left[\because y(\text{IF}) = \int Q(\text{IF})dx + C\right]$

$y(1 + \sin x) = \dfrac{-x^2}{2} + C$ **1**

Now, we have $y = 1$, when $x = 0$

or $\quad 1(1 + \sin 0) = -\dfrac{0}{2} + C$

or $\quad C = +1$ ½

Putting $C = 1$ in eqn. (ii), we get

$y(1 + \sin x) = -\dfrac{x^2}{2} + 1$

or $2y(1 + \sin x) + x^2 - 2 = 0$ **1**

Topic-4 Homogeneous Differential Equations

Concepts Covered • *Solution of Homogenous Differential Equation of first order and first degree*

Revision Notes

➢ **Homogeneous Differential Equations and their solution :**

➲ **Identifying a Homogeneous Differential equation :**

STEP 1 : Write down the given differential equation in the form $\dfrac{dy}{dx} = f(x, y)$.

STEP 2 : If $f(kx, ky) = k^n f(x, y)$, then the given differential equation is **homogeneous** of degree 'n'.

Key Words

Homogeneous: To be **Homogeneous** a function must pass this test:
$f(zx, zy) = z^n f(x, y)$

Key Word

In other words,
Homogeneous is when we can take a function: $f(x, y)$ multiply each variable by $z : f(zx, zy)$
and then can rearrange it to get this: $z^n f(x, y)$
e.g.: $x + 3y$
Start with: $f(x, y) = x + 3y$
Multiply each variable by $z : f(zx, zy) = zx + 3zy$
Let's rearrange it by factoring out z : $f(zx, zy) = z(x + 3y)$
And $x + 3y$ is $f(x, y)$: $f(zx, zy) = z\, f(x, y)$
which is what we wanted, with $n = 1$: $f(zx, zy) = z^1 f(x, y)$

➲ **Solving a Homogeneous Differential Equation:**

CASE I : If $\quad \dfrac{dy}{dx} = f(x, y)$

Put $\quad y = vx$

or $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

CASE II : If $\quad \dfrac{dx}{dy} = f(x, y)$

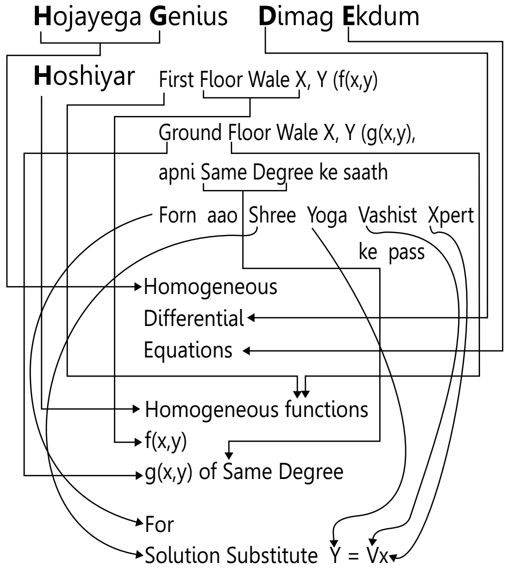
Put $\quad x = vy$

or $\quad \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

Then, we separate the variables to get the required solution.

Mnemonics

Homogeneous Differential Equation



Interpretation :
Differential equation can be expressed in the form $\dfrac{dy}{dx} = f(x, y)$ or $\dfrac{dx}{dy} = g(x, y)$ where $f(x,y)$ and $g(x,y)$ are homogeneous functions of sum is called a homogeneous Differential equation. These equations can be solved by substituting $y = vx$ so that dependent variable y is changed to another variable v, where v is some unknown function.

OBJECTIVE TYPE QUESTIONS

A Multiple Choice Questions

Q. 1. The differential equation of the form $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$ is called [U]

(A) linear differential equation
(B) partial differential equation
(C) homogeneous differential equation
(D) non-homogeneous differential equation

Ans. Option (C) is correct.

Explanation: The differential equation of the form $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$ or $\dfrac{dx}{dy} = g\left(\dfrac{x}{y}\right)$ is called a homogeneous differential equation.

Q. 2. A differential equation of the form $\dfrac{dy}{dx} = F(x, y)$ where $F(x, y)$ is a homogeneous function of degree zero. Differential equation of the form $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$ is a homogeneous differential equation of degree: [U]

(A) 0 **(B)** 1
(C) 2 **(D)** Not defined

Ans. Option (B) is correct.

Q. 3. A differential equation of the form $\dfrac{dx}{dy} = G(x, y)$ where $G(x, y)$ is a homogeneous function of degree zero. Differential equation of the form $\dfrac{dx}{dy} = g\left(\dfrac{x}{y}\right)$ is a homogeneous differential equation of order: [U]

(A) 0 **(B)** 2
(C) 1 **(D)** None of these

Ans. Option (C) is correct.

Q. 4. To solve the homogeneous differential equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, we put: U

(A) $x = vy$ (B) $y = vx$

(C) $x = v$ (D) $y = v$

Ans. Option (B) is correct.

Q. 5. To solve the homogeneous differential equation the form $\frac{dx}{dy} = g\left(\frac{x}{y}\right)$, we put A I

(A) $y = vx$ (B) $y = x$

(C) $y = v$ (D) $x = vy$

Ans. Option (D) is correct.

Q. 6. A function F(x, y) is said to be homogeneous function of degree n (a non-negative integer), if U

(A) $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ (B) $F(\lambda x, \lambda y) = \frac{1}{\lambda^n} F(x, y)$

(C) $F(\lambda x, \lambda y) = \lambda F(x, y)$ (D) None of these

Ans. Option (A) is correct.

Q. 7. A function F(x, y) is a homogeneous function of degree n, if A I

(A) $F(x, y) = x^n f\left(\frac{y}{x}\right)$ (B) $F(x, y) = y^n g\left(\frac{x}{y}\right)$

(C) Both (A) and (B) (D) $F(x, y) = x^{-n} f\left(\frac{y}{x}\right)$

Ans. Option (C) is correct.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. For what value of n is the following a homogeneous differential equation : $\frac{dy}{dx} = \frac{x^3 - y^n}{x^2y + xy^2}$

U [CBSE SQP 2020-21]

Sol. 3 1

[CBSE Marking Scheme 2020]

Detailed Solution:

Homogeneous differential equation must have same degree in both numerator and denominator, it means

$$\frac{dy}{dx} = \frac{x^3 - y^n}{x^2y + xy^2}$$

so, $n = 3$

Short Answer Type Questions-I (2 marks each)

Q. 1. Find the general solution of the following differential equation:

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$ [SQP 2021-2022]

Sol. We have the differential equation:

$$\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right)$$

The equation is a homogeneous differential equation,

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ 1

The differential equation becomes

$$v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \frac{dv}{\sin v} = -\frac{dx}{x}$$

$$\Rightarrow \operatorname{cosec} v\, dv = -\frac{dx}{x}$$ ½

Integrating both sides, we get

$\log|\operatorname{cosec} v - \cot v| = -\log|x| + \log k, k > 0$

(Here $\log|k|$ is constant an arbitrary)

$\Rightarrow \log|(\operatorname{cosec} v - \cot v)x| = \log k$ 1

$\Rightarrow |(\operatorname{cosec} v - \cot v)x| = k$

$\Rightarrow x(\operatorname{cosec} v - \cot v) = \pm k$

$\Rightarrow \left(\operatorname{cosec}\frac{y}{x} - \cot\frac{y}{x}\right)x = c,$

where is the required general solution ½

[CBSE Marking Scheme 2022]

Q. 2. Solve the differential equation :

$$x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} + x - y\sin\left(\frac{y}{x}\right) = 0$$

Given that $x = 1$ when $y = \frac{\pi}{2}$.

R&U [CBSE Delhi Set I, II-2020]

Q. 3. Find the general solution of the differential equation

$ye^{x/y}\, dx = (xe^{x/y} + y^2)\, dy, y \neq 0$

Sol. Given differential equation can be written as

$$\frac{dx}{dy} = \frac{xe^{x/y} + y^2}{ye^{x/y}}$$ ½

Put $\frac{x}{y} = v$

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$ 1

$$\Rightarrow v + y\frac{dv}{dy} = \frac{ve^v + y}{e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{y}{e^v}$$

$$\therefore \quad \int e^v\,dv = \int dy$$

$$\Rightarrow \quad e^v = y + C \qquad \textbf{1}$$

$$\Rightarrow \quad e^{x/y} = y + C, \qquad ½$$

which is the required solution

[CBSE Marking Scheme 2020] (Modified)

Detailed Solution:

Given $\quad ye^{x/y}dx = (xe^{x/y} + y^2)\,dy,\; y \neq 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ye^{x/y}}{xe^{x/y}+y^2}$$

$$\frac{dx}{dy} = \frac{xe^{x/y}+y^2}{ye^{x/y}}$$

Put $\quad x = vy$

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{vye^v + y^2}{ye^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{vye^v + y^2}{ye^v} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{vye^v + y^2 - vye^v}{ye^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{y^2}{ye^v}$$

$$\Rightarrow \quad \frac{dv}{dy} = \frac{1}{e^v}$$

$$\Rightarrow \quad \int e^v dv = \int dy + C$$

$$\Rightarrow \quad e^v = y + C$$

$\Rightarrow \quad e^{x/y} = y + C$ is the required solution

Q. 4. Solve the differential equation :

$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$, given that $y = 0$ when $x = 1$.

A I R&U **[CBSE Delhi Set-I, 2019]**

OR

Find the general solution of the differential equation:

$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

R&U **[NCERT][Comptt. Delhi-2016]**

Sol. Writing $\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x} = \dfrac{y}{x} + \sqrt{1+\left(\dfrac{y}{x}\right)^2}$

Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ½

Differential equation becomes

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \log\left|v+\sqrt{1+v^2}\right| = \log|x| + \log c \qquad ½$$

$$\Rightarrow v + \sqrt{1+v^2} = cx \Rightarrow y + \sqrt{x^2+y^2} = cx^2 \qquad \textbf{1}$$

when $x = 1, y = 0 \Rightarrow c = 1, [v = 0]$ ½

$$\therefore y + \sqrt{x^2+y^2} = x^2 \qquad ½$$

[CBSE Marking Scheme, 2019] (Modified)

Detailed Solution :

Given differential equation is

$$xdy - ydx = \sqrt{x^2+y^2}\;dx$$

$$\Rightarrow \quad xdy = \left(\sqrt{x^2+y^2} + y\right)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x} \qquad ...(i)$$

The given differential equation is homogenous with zero degree

So, $\quad$ put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

From eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2+(vx)^2}+vx}{x}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{\sqrt{x^2(1+v^2)}+vx}{x} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2} + v - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log\left|v+\sqrt{v^2+1}\right| = \log x + \log c$$

$$\Rightarrow \quad \log\left|v+\sqrt{v^2+1}\right| = \log(cx)$$

$$\Rightarrow \quad v + \sqrt{v^2+1} = cx$$

Put $\quad v = \dfrac{y}{x}$, we get

$$\frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 + 1} = cx$$

$$\Rightarrow \quad y + \sqrt{x^2+y^2} = cx^2 \qquad ...(ii)$$

Given, $\quad x = 1$ when $y = 0$

$$\therefore \quad 0 + \sqrt{1+0} = c \times 1$$

$$\Rightarrow \quad c = 1$$

Put $c = 1$ in eq (ii) we get

$$y + \sqrt{x^2+y^2} = x^2$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = x^2 - y$$

$$\Rightarrow \quad (x^2 + y^2) = (x^2 - y)^2$$

Q. 5. Find the particular solution of the differential equation

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

U [CBSE OD Set-I, 2019, Delhi Set-II, 2020]

(Give n that $y = \frac{\pi}{4}$ at $x = 1$)

Q. 6. Find the general solution of the differential equation:

$$(x - y)\frac{dy}{dx} = x + 2y.$$

A I R&U [OD 2017] [SQP 2017-18]

Sol. Given differential equation can be written as

$$\frac{dy}{dx} = \frac{x+2y}{x-y}$$

or $\quad v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$, where $y = vx$ **1**

or $\quad \frac{v-1}{v^2+v+1}dv = -\frac{1}{x}dx$

Integrating both sides, we get

$$\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv - \frac{3}{2}\int\frac{1}{v^2+v+1}dv = -\log|x| + C$$ ½

or $\frac{1}{2}\log|v^2+v+1| - \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log|x| + C$

or $\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right| - \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = -\log|x| + C$ **1**

or $\log|x^2 + xy + y^2| = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C_1$

where $\quad C_1 = 2C.$ ½

[CBSE Marking Scheme 2017] (Modified)

Q. 7. Find the particular solution of the differential equation $2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$ given that $x = 0$ when $y = 1$. A I R&U [Foreign 2017] [NCERT] [OD 2016]

Sol. $\quad \frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}}$

Put, $\quad \frac{x}{y} = v,$

then $\quad \frac{dx}{dy} = v + y\frac{dv}{dy}$ **1**

$$v + y\frac{dv}{dy} = \frac{2vye^v - y}{2ye^v}$$

$$2\int e^v dv = -\int\frac{dy}{y}$$ ½

General solution is:

$$2e^v = -\log|y| + C$$

or $\quad 2e^{x/y} = -\log|y| + C$ ½

Given, $\quad x = 0, y = 1 \Rightarrow C = 2$

Particular solution is

$$2e^{\frac{x}{y}} + \log|y| = 2$$ **1**

[CBSE Marking Scheme 2016] (Modified)

Detailed Solution :

$$2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$$

or $\quad \frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}}$ ...(i)

$\therefore$ It is a homogeneous differential equation

Put $\quad x = vy$

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Thus, $\quad v + y\frac{dv}{dy} = \frac{2e^v v - 1}{2e^v}$

or $\quad y\frac{dv}{dy} = \frac{2e^v\cdot v - 1}{2e^v} - v$

or $\quad y\frac{dv}{dy} = \frac{-1}{2e^v}$

or $\quad 2e^v dv = \frac{-1}{y}dy \quad y \neq 0$

or $\quad 2\int e^v dv = -\int\frac{1}{y}dy$

or $\quad 2e^v = -\log|y| + C$

or $\quad 2e^{x/y} + \log|y| = C$ ...(ii)

It is given that $x = 0$, when $y = 1$.

So, putting $x = 0$, $y = 1$ in eqn. (ii), we have

$$2e^0 + \log 1 = C$$

or $\quad C = 2$

Putting $C = 2$ in eqn. (ii), we have

$$2e^{x/y} + \log y = 2$$

Q. 8. Solve the following differential equation :

$$\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0.$$

R&U [NCERT][SQP 2016-17]

Sol. We have $\left(1+e^{\frac{x}{y}}\right)dx = \left(\frac{x}{y}-1\right)e^{\frac{x}{y}}\,dy$

or $$\frac{dx}{dy} = \frac{\left(\frac{x}{y}-1\right)e^{\frac{x}{y}}}{\left(1+e^{\frac{x}{y}}\right)} = f\left(\frac{x}{y}\right),$$

Hence, homogeneous differential equation ½

$$x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$v + y\frac{dv}{dy} = \frac{(v-1)e^v}{1+e^v} \qquad ½$$

$$\int \frac{1+e^v}{e^v+v}dv = -\int\frac{dy}{y}$$

$$\log_e|e^v + v| = -\log_e|y| + \log_e C \qquad 1$$

or $\log_e|(e^v + v)y| = \log_e C$ ½

or $(e^v + v)y = C = A$

or $\left(e^{\frac{x}{y}} + \frac{x}{y}\right)y = A$, the general solution. ½

[CBSE Marking Scheme 2016,] (Modified)

Q. 9. Solve the differential equation :

$$\mathbf{2ye^{\frac{x}{y}}dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0.}$$

R&U **[O.D. Set-II, 2016][O.D. Set-I, 2017]**

Topper Answer, 2017

Sol. $2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$

$\frac{dy}{dx} = \frac{-2ye^{x/y}}{y - 2xe^{x/y}}$

$\therefore v + y\frac{dv}{dy} = \frac{y - 2vye}{-2ye^{v}} = -\frac{1}{2}e^{-v} + v \Rightarrow y\frac{dv}{dy} = -\frac{1}{2}e^{-v}$

[illegible] $\frac{1}{2}\frac{dy}{y}$

Integrating

$e^{v} = -\frac{1}{2}\log y + C$

$\therefore e^{x/y} = -\frac{1}{2}\log y + C$ is the required solution to the differential equation

Q. 10. Find the solution of the differential equation

$$\boldsymbol{(xdy - ydx)\,y\sin\left(\frac{y}{x}\right) = (ydx + xdy)x\cos\left(\frac{y}{x}\right).}$$

A I R&U **[OD Comptt. 2013] [NCERT]**
[Outside Delhi Set I, II, III Comptt. 2016]

Sol. Given differential equation can be written as

$$(x\,dy - y\,dx)y\sin\left(\frac{y}{x}\right) = (ydx + xdy)x\cos\left(\frac{y}{x}\right)$$

$$xy\sin\left(\frac{y}{x}\right)dy - y^2\sin\left(\frac{y}{x}\right)dx = yx\cos\left(\frac{y}{x}\right)dx + x^2\cos\left(\frac{y}{x}\right)dy$$

$$xy\sin\left(\frac{y}{x}\right)\frac{dy}{dx} - y^2\sin\left(\frac{y}{x}\right) = xy\cos\left(\frac{y}{x}\right) + x^2\cos\left(\frac{y}{x}\right)\frac{dy}{dx}$$

or $$\frac{dy}{dx} = \frac{y^2\sin\left(\frac{y}{x}\right) + xy\cos\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)} \qquad 1$$

Put $(y/x) = v$ to get $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ $$v + x\frac{dv}{dx} = \frac{v^2\sin v + v\cos v}{v\sin v - \cos v} \qquad 1$$

or $$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

or $$\int\frac{\cos v - v\sin v}{v\cos v}dv = -2\int\frac{dx}{x} \qquad ½$$

or $\log|v\cos v| + \log x^2 = \log C$

or $x^2 v\cos v = C$ or $xy\cos(y/x) = C$ ½

[CBSE Marking Scheme 2016] (Modified)

Long Answer Type Questions (5 marks each)

Q. 1. Solve the differential equation $x^2dy + (xy + y^2) dx = 0$ given $y = 1$, when $x = 1$.

A I R&U [NCERT] [O.D. Comptt. 2015]

Sol. Given, $x^2dy + (xy + y^2)dx = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{-(xy+y^2)}{x^2} \quad \text{½}$$

Put $y = vx$

or $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ The differential equation becomes

$$v + x\frac{dv}{dx} = -(v + v^2) \quad \mathbf{1}$$

$$\text{or} \quad \frac{dv}{v^2+2v} = -\frac{dx}{x} \quad \text{½}$$

$$\text{or} \quad \int\frac{dv}{(v+1)^2-1^2} = -\int\frac{dx}{x}$$

$$\text{or} \quad \frac{1}{2}\log\frac{v}{v+2} = -\log x + \log C \quad \mathbf{1}$$

$$\text{or} \quad \frac{C}{x} = \sqrt{\frac{y}{y+2x}} \quad \mathbf{1}$$

If $x = 1, y = 1$, then $C = \frac{1}{\sqrt{3}}$

$$\text{or} \quad \frac{1}{\sqrt{3}x} = \sqrt{\frac{y}{y+2x}} \quad \mathbf{1}$$

[CBSE Marking Scheme 2015] (Modified)

Q. 2. Find the particular solution of the differential equation :

$$xe^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right) + x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = 0.$$

for $x = 1, y = 0$. R&U [SQP 2015]

Sol. Given differential equation is homogeneous.

$\therefore$ Putting $y = vx$ to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$ **1**

$$\frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - xe^{\frac{y}{x}}}{x\sin\left(\frac{y}{x}\right)}$$

$$\text{or} \quad v + x\frac{dv}{dx} = \frac{v\sin v - e^v}{\sin v} \quad \mathbf{1}$$

$$\text{or} \quad v + x\frac{dv}{dx} = v - \frac{e^v}{\sin v}$$

$$\text{or} \quad x\frac{dv}{dx} = -\frac{e^v}{\sin v}$$

$$\therefore \quad \int \sin v\, e^{-v} dv = -\int\frac{dx}{x}$$

$$\text{or} \quad I_1 = -\log x + C_1 \quad \text{....(i)}\ \mathbf{1}$$

$$\text{or} \quad I_1 = -\sin v.e^{-v} + \int \cos v\, e^{-v} dv$$

$$\text{or} \quad I_1 = -\sin v.e^{-v} - \cos v e^{-v} - \int \sin v.e^{-v} dv$$

$$\text{or} \quad I_1 = -\frac{1}{2}(\sin v + \cos v)e^{-v}$$

Putting (i), $(\sin v + \cos v)\, e^{-v} = \log x^2 + 2C_1$

$$\text{or} \left[\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right]e^{\frac{-y}{x}} = \log x^2 + C_2 \quad \mathbf{1}$$

For $x = 1, y = 0$ or $C_2 = 1$ ½

Hence, solution is

$$\left[\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right]e^{\frac{-y}{x}} = \log x^2 + 1 \quad \text{½}$$

[CBSE Marking Scheme, 2015] (Modified)

Q. 3. Show that the differential equation

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$$

is homogeneous. Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$. [O.D. Set I, II, III Comptt. 2015]

A I R&U [NCERT]

Q. 4. Show that the differential equation $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ is homogeneous and also solve it.

R&U [All India 2015]

Sol. Given differential equation is

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2} \quad \text{...(i)}$$

Let $F(x, y) = \frac{y^2}{xy - x^2}$

Now, on replacing x by λx and y by λy, we get

$$F(\lambda x, \lambda y) = \frac{\lambda^2 y^2}{\lambda^2(xy - x^2)} = \lambda^0 \frac{y^2}{xy - x^2} = \lambda^0 F(x, y)$$

Thus, the given differential equation is a homogeneous differential equation. **1**

Now, to solve it, put $y = vx$

$$\text{or} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{vx^2 - x^2} = \frac{v^2}{v-1} \quad \mathbf{1}$$

$$\text{or} \quad x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1}$$

$$\text{or} \quad x\frac{dv}{dx} = \frac{v}{v-1} \quad \text{or} \quad \frac{v-1}{v}dv = \frac{dx}{x} \quad \mathbf{1}$$

On integrating both sides, we get

$$\int\left(1-\frac{1}{v}\right)dv = \int\frac{dv}{x}$$

or $\quad v - \log|v| = \log|x| + C$

or $\quad \frac{y}{x} - \log\left|\frac{y}{x}\right| = \log|x| + C \left[\text{put } v = \frac{y}{x}\right]$ **1**

or $\quad \frac{y}{x} - \log|y| + \log|x| = \log|x| + C$

$$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

$\therefore \quad \frac{y}{x} - \log|y| = C$

which is the required solution. **1**

[CBSE Marking Scheme, 2015] (Modified)

CASE BASED QUESTIONS

Case based MCQs (1 mark each)

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

A Veterinary doctor is examining a sick cat brought by a pet lover. When it was brought to the hospital, it was already dead. The pet lover wanted to find its time of death. He took the temperature of the cat at 11.30 pm which was 94.6°F. He took the temperature again after one hour; the temperature was lower than the first observation. It was 93.4°F. The room in which the cat was put is always at 70°F. The normal temperature of the cat is taken as 98.6°F when it was alive. The doctor estimated the time of death using Newton law of cooling which is governed by the differential equation: $\frac{dT}{dt} \propto (T-70)$, where 70°F is the room temperature and T is the temperature of the object at time t.

Substituting the two different observations of T and t made, in the solution of the differential equation $\frac{dT}{dt} = k(T-70)$ where k is a constant of proportion, time of death is calculated. **[CBSE QB-2021]**

Q. 1. What will be the degree of the above given differential equation?

(A) 2 (B) 1
(C) 0 (D) 3

Ans. Option (B) is correct.

Q. 2. Which method of solving a differential equation helped in calculation of the time of death?

(A) Variable separable method
(B) Solving Homogeneous differential equation
(C) Solving Linear differential equation
(D) all of the above

Ans. Option (A) is correct.

Q. 3. If the temperature was measured 2 hours after 11.30 pm, what will be the change in time of death?

(A) No change
(B) Death time increased
(C) Death time decreased
(D) Death time always constant

Ans. Option (A) is correct.

Q. 4. The solution of the differential equation $\frac{dT}{dt} = k(T-70)$ is given by,

(A) $\log|T-70| = kt + C$
(B) $\log|T-70| = \log|kt| + C$
(C) $T-70 = kt + C$
(D) $T-70 = kt\,C$

Ans. Option (A) is correct.

Explanation:

$dT - k(T-70)\,dt = 0$

This is separable

$$\frac{dT}{dt} = k(T-70)$$

$$\left(\frac{1}{T-70}\right)\frac{dT}{dt} = k$$

$$\int\left(\frac{1}{T-70}\right)\frac{dT}{dt}\,dt = k\int dt$$

$$\int\left(\frac{1}{T-70}\right)dT = k\int dt$$

$$\log|T-70| = kt + C$$

Q. 5. If $t = 0$ when T is 72, then the value of C is

(A) –2 (B) 0
(C) 2 (D) log 2

Ans. Option (D) is correct.

II. Read the following text and answer the following questions on the basis of the same:

Polio drops are delivered to 50K children in a district. The rate at which polio drops are given is directly proportional to the number of children who have not been administered the drops. By the end of 2nd week half the children have been given the polio drops. How many will have been given the drops by the end of 3rd week can be estimated using the solution to the differential equation $\frac{dy}{dx} = k(50-y)$ where x denotes the number of weeks and y the number of children who have been given the drops. **[CBSE QB-2021]**

Q. 1. State the order of the above given differential equation.

(A) 2 (B) 1
(C) 0 (D) Can't define

Ans. Option (B) is correct.

Q. 2. Which method of solving a differential equation can be used to solve $\frac{dy}{dx} = k(50 - y)$?

(A) Variable separable method

(B) Solving Homogeneous differential equation

(C) Solving Linear differential equation

(D) All of the above

Ans. Option (A) is correct.

Q. 3. The solution of the differential equation $\frac{dy}{dx} = k(50 - y)$ is given by,

(A) $\log|50 - y| = kx + C$

(B) $-\log|50 - y| = kx + C$

(C) $\log|50 - y| = \log|kx| + C$

(D) $50 - y = kx + C$

Ans. Option (B) is correct.

Explanation:

$$\frac{dy}{dx} = k(50 - y)$$

$$\int \frac{dy}{50 - y} = \int k\,dx$$

$$-\log|50 - y| = kx + C$$

Q. 4. The value of C in the particular solution given that $y(0) = 0$ and $k = 0.049$ is

(A) log 50 **(B)** $\log\frac{1}{50}$

(C) 50 **(D)** –50

Ans. Option (B) is correct.

Explanation:

Given, $y(0) = 0$ and $k = 0.049$

We have,

$$-\log|50 - y| = kx + C$$

$$\log|50 - y| = -kx - C$$

$$\log|50 - 0| = 0 - C$$

$$[\because x = 0, K = 0.049, y(0) = 0]$$

$$\log 50 = -C$$

$$C = \log\frac{1}{50}$$

Q. 5. Which of the following solutions may be used to find the number of children who have been given the polio drops?

(A) $y = 50 - e^{kx}$ **(B)** $y = 50 - e^{-kx}$

(C) $y = 50(1 - e^{-kx})$ **(D)** $y = 50(e^{-kx} - 1)$

Ans. Option (C) is correct.

Explanation: We have

$$-\log|50 - y| = kx + C$$

$$-\log|50 - y| = kx + \log\frac{1}{50}$$

$$\log\frac{50 - y}{50} = -kx$$

$$\frac{50 - y}{50} = e^{-kx}$$

$$50 - y = 50e^{-kx}$$

$$y = 50 - 50e^{-kx}$$

$$y = 50(1 - e^{-kx})$$

Case based Subjective Questions (2 mark each)

I. Read the following text and answer the following questions on the basis of the same:

Reeta note down the following about homogeneous differential equations in her note book,in mathematics class.

If the equation is of form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

or $$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

where f(x, y), g(x, y) are homogeneous functions of the same degree in x and y, then put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So, that the dependent variable y changed to another variable v and then apply variable separable method.

Q. 1. Solve the differential equations:

$(x - \sqrt{xy})dy = ydx$

Sol. Given $(x - \sqrt{xy})dy = ydx$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x - \sqrt{xy}} \quad \text{...(i)}$$

Dividing Nr and Dr. of RHS of (1) by x, we get

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{1 - \sqrt{\frac{y}{x}}},$$

which is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

Therefore, (i) is a homogeneous differential equation,

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad ½$$

From (i), we get

$$v + x\frac{dv}{dx} = \frac{v}{1 - \sqrt{v}}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^{\frac{3}{2}}}{1 - \sqrt{v}}$$

$$\Rightarrow \quad \frac{1 - \sqrt{v}}{v^{\frac{3}{2}}}dv = \frac{dx}{x}$$

$$\Rightarrow \quad \left(v^{-\frac{3}{2}} - \frac{1}{v}\right)dv = \frac{dx}{x} \quad ½$$

Integrating both sides, we get

$$\frac{v^{-\frac{1}{2}}}{-\frac{1}{2}} - \log|v| = \log|x| + c$$

$$\Rightarrow \quad \frac{-2}{\sqrt{v}} - \log|vx| = -c$$

$$\Rightarrow \quad 2\sqrt{\frac{x}{y}} + \log|y| = -c = \text{A (say)} \qquad \mathbf{1}$$

Hence, solution is $2\sqrt{\frac{x}{y}} + \log|y| = \text{A}$, A is arbitrary constant.

Q. 2. Solve the differential equations: **2**

$$2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$$

Sol. Given, $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ ...(i)

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \tfrac{1}{2}$$

Substituting the values of y and $\frac{dy}{dx}$ in eq. (i), we get

$$2\left(v + x\frac{dv}{dx}\right) = v + v^2$$

$$\Rightarrow \quad 2x\frac{dv}{dx} = v^2 - v$$

$$\Rightarrow \quad \frac{2}{v^2 - v}dv = \frac{dx}{x}$$

$$\Rightarrow \quad 2\left(\frac{1}{v-1} - \frac{1}{v}\right)dv = \frac{dx}{x} \qquad \tfrac{1}{2}$$

Integrating both sides, we get

$2(\log|v-1| - \log|v|) = \log|x| + c$

$$\Rightarrow \quad 2\log\left|\frac{v-1}{v}\right| = \log|x| + c$$

$\Rightarrow 2\log\left|\frac{y-x}{y}\right| = \log|x| + c$, c is arbitrary constant **1**

Solutions for Practice Questions (Topic-1)

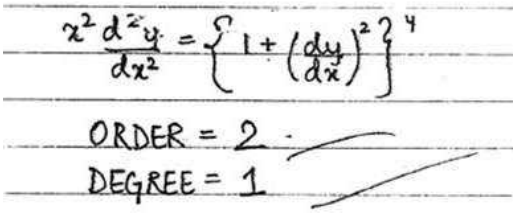
Very Short Answer Type Questions

2. Order = 2, degree = 1 ½ + ½

[CBSE Marking Scheme, 2019]

Topper Answer, 2019

Sol.



Commonly Made Error

- Some students write the degree as 4 or 8 considering it as the highest power.

Answering Tip

- Degree is the highest power of the highest order derivative.

3. Order = 2, degree = 2 ½+ ½

[CBSE Marking Scheme 2019]

Detailed Solution :

Given, $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$

The highest order derivative is $\left(\frac{d^2y}{dx^2}\right)$, hence the order of given differential equation is 2.

Also, the power of the highest order derivative is 2, hence the degree of differential equation is 2.

Therefore, Order = 2 and Degree = 2

Commonly Made Error

- Some students take the degree as the highest power of the derivative.

Answering Tip

- Degree is the highest power of the higher order derivative.

Short Answer Type Questions-I

4. Here, $\left\{\frac{d^2y}{dx^2} + (1+x)\right\}^3 = -\frac{dy}{dx}$ **1**

Thus, order is 2 and degree is 3. So, the sum is 5 **1**

[CBSE Marking Scheme 2016]

Solutions for Practice Questions (Topic-2)

Very Short Answer Type Questions

1. 0

[CBSE SQP Marking Scheme 2020-21]

3.
$$\int \frac{dy}{e^{-2y}} = \int x^3 dx$$
$$\int e^{2y} dy = \int x^3 dx \quad ½$$
or $$\frac{e^{2y}}{2} = \frac{x^4}{4} + C$$
or $$\frac{1}{2}e^{2y} = \frac{x^4}{4} + C \quad ½$$
$$2e^{2y} = x^4 + C_1$$
where $(C_1 = 4C)$

[CBSE Marking Scheme 2015]

Short Answer Type Questions-I

3. For $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$ ½

Integrating, we get

$\tan^{-1} y = \tan^{-1} x + C.$

As $x = 0, y = \sqrt{3}$ so $\tan^{-1}\sqrt{3} = C$ or $C = \frac{\pi}{3}$ ½

Solution is $\tan^{-1} y = \tan^{-1} x + \frac{\pi}{3}$ ½

$$\tan^{-1} \frac{y-x}{1+xy} = \frac{\pi}{3}$$
$$\frac{y-x}{1+xy} = \tan\frac{\pi}{3} = \sqrt{3}$$
$$y - x = \sqrt{3}\,(1 + xy) \quad ½$$

[CBSE Marking Scheme 2017]

Short Answer Type Questions-II

Sol. Given equation can be written as

$$\int \frac{dy}{2e^{-y}-1} = \int \frac{dx}{x+1}$$
$$\Rightarrow \int \frac{e^y}{2-e^y} dy = \int \frac{dx}{x+1} \quad ½$$
$$\Rightarrow -\log|2-e^y| + \log c = \log|x+1| \quad 1$$
$$\Rightarrow (2-e^y)(x+1) = c$$

When $x = 0, y = 0 \Rightarrow c = 1$ 1

$\therefore$ Solution is $(2 - e^y)(x + 1) = 1$ ½

[CBSE Marking Scheme, 2020] (Modified)

Solutions for Practice Questions (Topic-3)

Very Short Answer Type Questions

1. Writing the given equation as

$$\frac{dy}{dx} + \frac{1}{\sqrt{x}} y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Here, $P = \frac{1}{\sqrt{x}}$

$\therefore$ $I.F. = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}} dx}$ ½

$I.F. = e^{2\sqrt{x}}$ ½

[CBSE Marking Scheme 2015]

Short Answer Type Questions-II

3. Given differential equation can be written as

$$\frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = \frac{1}{x} \quad 1$$

Getting integrating factor $= e^{x - \log x}$ or $\frac{e^x}{x}$ 1

[CBSE Marking Scheme 2017]

Short Answer Type Questions-II

7. Integrating factor is $e^{\int 2\tan x\,dx}$

$= e^{2\log \sec x} = \sec^2 x$ ½

$y\sec^2 x = \int \sin x \cdot \sec^2 x\,dx$ 1

or $y\sec^2 x = \int \sec x \tan x\,dx$

or $y\sec^2 x = \sec x + C$ ½

$x = \frac{\pi}{3}, y = 0;\ 0 = 2 + C$

or $C = -2$ ½

$\therefore$ $y\sec^2 x = \sec x - 2$

or $y = \cos x - 2\cos^2 x$ ½

[CBSE Marking Scheme 2018] (Modified)

10. Given differential equation can be written as

$$(1+x^2)\frac{dy}{dx} + y = \tan^{-1} x$$
$$\Rightarrow \frac{dy}{dx} + \frac{1}{1+x^2} y = \frac{\tan^{-1} x}{1+x^2} \quad 1$$

Integrating factor $= e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$ ½

$\therefore$ Solution is $y.e^{\tan^{-1}x} = \int \tan^{-1}x.e^{\tan^{-1}x}\frac{1}{1+x^2}dx$ ½

$\Rightarrow \quad ye^{\tan^{-1}x} = e^{\tan^{-1}x}.(\tan^{-1}x-1)+c$ **1**

or $\quad y = (\tan^{-1}x-1)+c.e^{-\tan^{-1}x}$

[CBSE Marking Scheme, 2017] (Modified)

Long Answer Type Questions

2. The given equation can be written as

$$\frac{dy}{dx}+\frac{y}{x} = \cos x+\frac{\sin x}{x}$$ **1**

$$I.F. = e^{\int\frac{1}{x}dx} = e^{\log x} = x$$ **1**

$\therefore$ Solution is

$$y.x = \int(x\cos x+\sin x)dx+c$$ **1**

$\therefore \quad y.x = x\sin x+c$ **1**

or $\quad y = \sin x+\frac{c}{x}$

when $\quad x = \frac{\pi}{2}, y = 1$, we get $c = 0$

Required solution is $y = \sin x$ **1**

[CBSE Marking Scheme, 2017] (Modified)

Solutions for Practice Questions (Topic-4)

Short Answer Type Questions-II

2. Given differential equation gives

$$\frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right)-x}{x\sin\left(\frac{y}{x}\right)}$$ ½

$$\frac{y}{x} = v$$

$\Rightarrow \quad y = vx$

and $\quad \frac{dy}{dx} = v+x\frac{dv}{dx}$ ½

$\therefore \quad v+x\frac{dv}{dx} = \frac{v\sin v-1}{\sin v}$ ½

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-1}{\sin v}$

$\Rightarrow \quad \int\sin v\,dv = \int\frac{-1}{x}dx$ ½

$\therefore \quad -\cos v = -\log|x|+C$

or $\quad \cos\left(\frac{y}{x}\right) = \log|x|-C$ ½

Given $\quad x = 1$

when $\quad y = \frac{\pi}{2}$

$\Rightarrow \quad C = 0$

$\therefore \quad \cos\left(\frac{y}{x}\right) = \log|x|$ is the required solution ½

[CBSE Marking Scheme 2020]

Detailed Solution:

We have,

$$x\sin\left(\frac{y}{x}\right)\frac{dy}{dx}+x-y\sin\frac{y}{x} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right)-x}{x\sin\left(\frac{y}{x}\right)} \quad ...(i)$$

Above differential equation is a homogeneous equation

Put $\quad y = vx$

Then, $\quad \frac{dy}{dx} = v+x\frac{dv}{dx} \quad ...(ii)$

From (i) and (ii),

$$\Rightarrow \quad v+x\frac{dv}{dx} = \frac{vx.\sin\left(\frac{vx}{x}\right)-x}{x\sin\left(\frac{vx}{x}\right)}$$

$$\Rightarrow \quad v+x\frac{dv}{dx} = \frac{x(v\sin v-1)}{x\sin v}$$

$$\Rightarrow \quad v+x\frac{dv}{dx} = \frac{v\sin v-1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v-1}{\sin v}-v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v-1-v\sin v}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Rightarrow \quad \sin v\,dv = -\frac{1}{x}dx \quad [\text{Here } x\neq 0]$$

Now, integrating both sides

$$\Rightarrow \quad \int\sin v\,dv = -\int\frac{1}{x}dx$$

$$\Rightarrow \quad -\cos v = -\log|x|+C$$

Put, $\quad v = \frac{y}{x}$

$$\Rightarrow \quad -\cos\left(\frac{y}{x}\right) = -\log|x|+C \quad ...(iii)$$

Also, given that $x = 1$, when $y = \frac{\pi}{2}$

Put $x = 1$ and $y = \frac{\pi}{2}$ in (iii)

$$\Rightarrow \quad -\cos\left(\frac{\pi}{2}\right) = -\log 1+C$$

$$C = 0$$

$$\Rightarrow \quad -\cos\left(\frac{y}{x}\right) + \log|x| = 0$$

Therefore, $\log|x| = \cos\left(\frac{y}{x}\right)$ is the required solution.

5. The given differential equation can be written as:

Put $\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$ ½

$\frac{y}{x} = v$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, to get ½

$$v + x\frac{dv}{dx} = v - \tan v$$

$$\Rightarrow x\frac{dv}{dx} = -\tan v$$

$$\Rightarrow \cot v\, dv = -\frac{1}{x}dx,$$ ½

Integrating both sides we get,

$$\left.\begin{aligned} &\log|\sin v| = -\log|x| + \log c \\ &\Rightarrow \log|\sin v| = \log\left|\frac{c}{x}\right| \end{aligned}\right\}$$ ½

$\therefore$ Solution of differential equation is

$$\sin\left(\frac{y}{x}\right) = \frac{c}{x} \text{ or } x\cdot\sin\left(\frac{y}{x}\right) = c$$ ½

Put $x = 1$ and $y = \frac{\pi}{4}$, we get

$$1.\sin\left(\frac{\pi}{4}\right) = c$$

$$\Rightarrow c = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$\therefore$ Particular Solution is

$$x.\sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$$ ½

OR

Topper Answer, 2020

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) = f(y/x) \quad -(i)$$

$\therefore$ It is a homogeneous function.

let $\frac{y}{x} = V \Rightarrow y = Vx$

Differentiating with respect to x

$$\Rightarrow \frac{dy}{dx} = V + \frac{x\,dV}{dx}$$

$\therefore$ equation (i) can be written as

$$V + x\frac{dV}{dx} = V - \tan V$$

$$\Rightarrow \frac{x\,dV}{dx} = -\tan V$$

$$\Rightarrow \int -\cot V\, dV = \int \frac{dx}{x}$$

$\Rightarrow -\log\sin V = \log x + \log c$, $-\log c$ is integration constant.

$$\log\sin V + \log x = \log c$$

[2]

At $x=1, y=\pi/4$

$\sin\left(\frac{\pi}{4}\right) = C = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x\sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$ Answer

Long Answer Type Questions

3. $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$

or $\quad \frac{dy}{dx} = -\frac{x\sin^2\frac{y}{x} - y}{x}$ ½

$= \frac{y - x\sin^2\frac{y}{x}}{x}$

$= \frac{y}{x} - \sin^2\frac{y}{x}$ ...(i) ½

which is homogeneous.

Put, $\quad y = vx$

or $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

or $\quad v + x\frac{dv}{dx} = v - \sin^2 v$ [from (ii)]

or $\quad x\frac{dv}{dx} = -\sin^2 v$ ½

or $\quad \frac{dv}{\sin^2 v} = -\frac{dx}{x}$ ½

On integrating it, we get

$\int\frac{dv}{\sin^2 v} = -\int\frac{dx}{x}$

or $\quad \int \text{cosec}^2 v\,dv = -\int\frac{dx}{x}$

or $\quad -\cot v = -\log x + C$ ½

or $\quad \log x - \cot v = C$

Putting, $\quad v = \frac{y}{x}$

So general solution is : ½

$\log x - \cot\frac{y}{x} = C$

Putting, $\quad x = 1, y = \frac{\pi}{4}$

$\therefore \quad \log 1 - \cot\frac{\pi}{4} = C$ or $0 - 1 = C$

or $\quad C = -1$ **1**

$\therefore$ Particular solution is

$\log x - \cot\frac{y}{x} = -1$

or $\quad \cot\frac{y}{x} - \log x = 1.$ **1**

[CBSE Marking Scheme 2015] (Modified)

REFLECTIONS

(a) Differential equations are applied in various real time problem.

(b) General solution of differential equation contains as many arbitrary constants as the order of the differential equation.

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SELF ASSESSMENT PAPER - 03

Time: 1 hour **MM: 30**

UNIT-III

(A) OBJECTIVE TYPE QUESTIONS:

I. Multiple Choice Questions **[1×6 = 6]**

Q. **1.** The value of $\int \frac{\cot x \tan x}{\sec^2 x - 1} dx$ is

(A) $\cot x - x + c$ (B) $-\cot x + x + c$

(C) $\cot x + x + c$ **(D)** $-\cot x - x + c$

Q. 2. The value of $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx$ is

(A) $\tan x + \cot x + c$ (B) $\tan x - \cot x + c$

(C) $\text{cosec}\, x - \cot x + c$ **(D)** $\sec x - \text{cosec}\, x + c$

Q. 3. The solution of $\int a^x da$ is

(A) $\frac{a^x}{\log_e a} + c$ **(B)** $a^x \log_e a + c$ **(C)** $\frac{a^{x+1}}{x+1} + c$ **(D)** $xa^{x-1} + c$

Q. 4. The value of $\int \frac{(1+\log x)^2}{x} dx$ is

(A) $(1 + \log x)^3 + c$ **(B)** $3(1 + \log x)^3 + c$ **(C)** $\frac{1}{3}(1 + \log x)^3 + c$ **(D)** None of these

Q. 5. Area bounded by parabola $y^2 = x$ and straight line $2y = x$ is _______ sq. units.

(A) $\frac{4}{3}$ **(B)** 1 **(C)** $\frac{2}{3}$ **(D)** $\frac{1}{3}$

Q. 6. The solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ is

(A) $1 + xy + c(y + x) = 0$ **(B)** $x + y = c(1 - xy)$

(C) $y - x = c(1 + xy)$ **(D)** $1 + xy = c(x + y)$

II. Case-Based MCQs **[1×4 = 4]**

Attempt any 4 sub-parts from each questions. Each question carries 1 mark.

Read the following text and answer the following questions on the basis of the same.

Whatsapp became the world's most popular messaging application by 2015. As of 2021, WhatsApp is most popular global mobile messenger app worldwide with approximately two billion monthly active users. WhatsApp has also been criticized for spreading of fake news.

In a population of 5000 people, a rumour on whatsapp spreads at a rate proportional to the product of the number of people who have heard it and the number of people who have not. Also, it is given that 100 people intiate the rumour and a total of 500 people know the rumour after 2 days.

Q. 7. Find the maximum value of $y(t)$, if $y(t)$ denote the number of people who know the rumour at an instant t.

(A) 500 **(B)** 100 **(C)** 5000 **(D)** none of these

Q. 8. $\frac{dy}{dt}$ = ...

(A) $(y - 5000)$ **(B)** $y(y - 5000)$ **(C)** $y(500 - y)$ **(D)** $y(5000 - y)$

Q. 9. $y(0)$ = ...

(A) 100 **(B)** 500 **(C)** 600 **(D)** 200

Q. 10. $y(2)$ = ...

(A) 100 **(B)** 500 **(C)** 600 **(D)** 200

Q. 11. At any time t, the value of y is:

(A) $y = \frac{5000}{(e^{-5000kt} + 1)}$ **(B)** $y = \frac{5000}{(1 + e^{-5000kt})}$ **(C)** $y = \frac{5000}{(49e^{-5000kt} + 1)}$ **(D)** $y = \frac{5000}{49(1 + e^{-5000kt})}$

(B) SUBJECTIVE TYPE QUESTIONS:

III. Very Short Answer Type Questions **[1×3 = 3]**

Q. 12. Find: $\int \frac{3x}{3x - 1} dx$.

Q. 13. Write the general solution of differential equation $\frac{dy}{dx} = e^{x + y}$.

Q. 14. Using integration, Find the area bounded by the function $f(x) = x^3$, the X-axis and the lines $x = -1$.

IV. Short Answer Type Questions–I **[2×3 = 6]**

Q. 15. Find the value of $\int_0^1 x(1 - x)^n dx$.

Q. 16. Find : $\int \frac{x \cos^{-1} x}{\sqrt{1 - x^2}} dx$.

Q. 17. Find the integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1 + y}{x}$.

V. Short Answer Type Questions-II **[3×2 = 6]**

Q. 18. Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Q. 19. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx$.

VI. Long Answer Type Questions [1×5 = 5]

Q. 20. Solve the differential equation:

$$\frac{dy}{dx} - 3y \cot x = \sin 2x \text{ given } y = 2, \text{ when } x = \frac{\pi}{2}.$$

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