CHAPTER 7

INTEGRALS

Syllabus

Integration as inverse process of differentiation. Integration of a variety of functions by substitution; by partial fractions and by parts. Evaluation of simple integrals of the following types and problems based on them.

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{px + q}{ax^2 + bx + c} dx,$$

$$\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx, \int \sqrt{a^2 \pm x^2} dx, \int \sqrt{x^2 - a^2} dx, \int \sqrt{ax^2 + bx + c}\, dx$$

Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

In this chapter you will study

- *Various methods of integration and how to apply them to definite and indefinite integrals.*
- *The evaluation of integrals of rational function by partial fraction.*
- *How integral calculus is based on the idea of differential calculus learnt earlier.*

List of Topics

Topic-1 Indefinite Integral

Concepts Covered • *Meaning of Integral of function* • *Integration by Substitution* • *Integration by partial fraction* • *Integration by parts* • *Formulae for indefinite Integral*

Revision Notes

➢ **Meaning of Integral of Function**

If differentiation of a function $F(x)$ is $f(x)$ *i.e.*, if $\frac{d}{dx}[F(x)] = f(x)$, then we say that one integral or primitive or anti-derivative of $f(x)$ is $F(x)$ and in symbols, we write, $\int f(x)dx = F(x) + C$.

Therefore, we can say that integration is the inverse process of differentiation.

➢ **Methods of Integration**

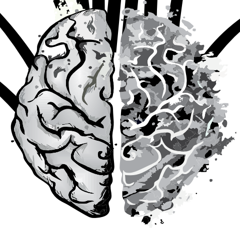
(a) Integration by Substitution Method :

In this method, we change the integral $\int f(x)dx$, where independent variable is x, to another integral in which independent variable is t (say) different from x such that x and t are related by $x = g(t)$.

Let $u = \int f(x)dx$ then, $\frac{du}{dx} = f(x)$

Again as $x = g(t)$ so we have $\frac{dx}{dt} = g'(t)$

Integrals



Example

$$\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx = 2\int_0^{\pi/4} \sin^2 x\,dx = 2\int_0^{\pi/4}\left(\frac{1-\cos 2x}{2}\right)dx = \int_0^{\pi/4}(1-\cos 2x)\,dx = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\pi}{4} - \frac{1}{2}$$

Integration

It is the inverse of differentiation. Let, $\frac{d}{dx}F(x) = f(x)$. Then, $\int f(x)dx = F(x)+c$, 'c': constant of integral. These integrals are called indefinite or general integrals.
Properties of indefinite integrals are

(i) $\int [f(x) \pm g(x)]\,dx = \int f(x)dx \pm \int g(x)dx$, **(ii)** $\int kf(x)dx = k\int f(x)dx$,

eg : $\int (3x^2 + 2x)dx = x^3 + x^2 + c$, where c is real.

Integration by substitution

The method in which we change the variable to some other variable is called the method of substitution. Below problems can be solved by substitution.

$\int \tan x dx = \log|\sec x| + c$ $\int \cot x dx = \log|\sin x| + c$

$\int \sec x dx = \log|\sec x + \tan x| + c$ $\int \text{cosec}\, x dx = \log|\text{cosec} x - \cot x| + c$.

Integration of some special functions

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ (ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

(iii) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ (iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

(v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$ (vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$.

(vii) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$.

(viii) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c$.

(ix) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$.

Integration by parts

$$\int f_1(x)f_2(x)dx = f_1(x)\int f_2(x)dx - \int\left[\frac{d}{dx}f_1(x)\int f_2(x)dx\right]dx$$

First fundamental theorem of integral calculus

Let the area function be defined by $A(x) = \int_a^x f(x)dx \,\forall x \ge a$,
where f is continuous on $[a,b]$
then $A'(x) = f(x) \,\forall x \in [a,b]$.

Definite integral as the limit of a sum

Some standard integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \ne -1$ like, $\int dx = x + c$

(ii) $\int \cos x\,dx = \sin x + c$ (iii) $\int \sin x\,dx = -\cos x + c$

(iv) $\int \sec^2 x\,dx = \tan x + c$ (v) $\int \text{cosec}^2 x\,dx = -\cot x + c$

(vi) $\int \sec x \tan x\,dx = \sec x + c$ (vii) $\int \text{cosec}\, x \cot x\,dx = -\text{cosec}\, x + c$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$ (ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1}x + c$

(x) $\int \frac{dx}{1+x^2} = \tan^{-1}x + c$ (xi) $\int \frac{dx}{1+x^2} = -\cot^{-1}x + c$

(xii) $\int e^x dx = e^x + c$ (xiii) $\int a^x dx = \frac{a^x}{\log a} + c$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c$ (xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\text{cosec}^{-1}x \quad c$

(xvi) $\int \frac{1}{x}dx = \log|x| + c$

Integration by partial fractions

A rational function of the form $\frac{P(x)}{Q(x)}[Q(x) \ne 0] = T(x) + \frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of $Q(x)$. We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms –

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, $a, b \ne 0$.

(ii) $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$ (iii) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)} + \frac{C}{x \quad b}$

(iv) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

(v) $\frac{Px+q}{ax^2+bx+c} = \frac{A\frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$

Second fundamental theorem of integral calculus

Let f be a continuous function of x defined on $[a, b]$ and let F be another function such that $\frac{d}{dx}F(x) = f(x) \,\forall x \in$ domain of f_1 then $\int_a^b f(x)dx = [F(x)+c]_a^b = F(b) - F(a)$. This is called the definite integral of f over the range $[a, b]$, where a and b are called the limits of integration, a being the lower limit and b be the upper limit.

Trace the Mind Map
- First Level
- Second Level
- Third Level

Now $\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} = f(x) \cdot g'(t)$

On integrating both sides w.r.t. t, we get

$$\int \left(\frac{du}{dt}\right) dt = \int f(x) g'(t) dt$$

or $$u = \int f[g(t)] g'(t) dt$$

i.e., $$\int f(x) dx = \int f[g(t)] g'(t) dt, \text{ where } x = g(t).$$

So, it is clear that substituting $x = g(t)$ in $\int f(x)$ will give us the same result as obtained by putting $g(t)$ in place of x and $g'(t)dt$ in place of dx.

(b) Integration by Partial Fractions:

Consider $\frac{f(x)}{g(x)}$ defines a rational polynomial function.

- If the degree of numerator *i.e.*, $f(x)$ is **greater than or equal to** the degree of denominator *i.e.*, $g(x)$ then, this type of rational function is called an **improper rational function**. And if degree of $f(x)$ is **smaller than** the degree of denominator *i.e.*, $g(x)$, then this type of rational function is called a **proper rational function**.
- In rational polynomial function if the degree (*i.e.*, highest power of the variable) of numerator (Nr.) is **greater than or equal to** the degree of denominator (Dr.), then (without any doubt) **always perform the division** *i.e.*, divide the **Nr.** by **Dr.** before doing anything and thereafter use the following:

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Table Demonstrating Partial Fractions or Various Forms

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where $x^2 + bx + c$ can't be factorized further.	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

(c) Integration by Parts :

If U and V be two functions of x, then

$$\int \underset{(I)}{U} \cdot \underset{(II)}{V} \, dx = U \int V dx - \int \left\{ \frac{dU}{dx} \int V dx \right\} dx$$

Key Formulae

Formulae for Indefinite Integrals

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(b) $\int \frac{1}{x} dx = \log|x| + C$

(c) $\int a^x dx = \frac{1}{\log a} a^x + C$

(d) $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$

(e) $\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + C$

(f) $\int \cos(ax) dx = \frac{1}{a}\sin(ax) + C$

(g) $\int \tan x dx = \log|\sec x| + C$ **or** $-\log|\cos x| + C$

(h) $\int \cot x dx = \log|\sin x| + C$ **or** $-\log|\text{cosec } x| + C$

(i) $\int \sec x dx = \log|\sec x + \tan x| + C$ **or** $\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(j) $\int \text{cosec } x dx = \log|\text{cosec } x - \cot x| + C$ **or** $\log\left|\tan\frac{x}{2}\right| + C$

(k) $\int \sec^2 x dx = \tan x + C$

(l) $\int \text{cosec}^2 x dx = -\cot x + C$

(m) $\int \sec x.\tan x dx = \sec x + C$

(n) $\int \text{cosec} x.\cot x dx = -\text{cosec} x + C$

(o) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

(p) $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(q) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(r) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(s) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x+\sqrt{x^2-a^2}\right| + C$

(t) $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x+\sqrt{x^2+a^2}\right| + C$

(u) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(v) $\int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax+b| + C$

(w) $\int \lambda dx = \lambda x + C$, where 'λ' is a constant.

(x) $\int \sqrt{x^2-a^2} dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

(y) $\int \sqrt{x^2+a^2} dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

(z) $\int \sqrt{a^2-x^2} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

OBJECTIVE TYPE QUESTIONS

A Multiple Choice Questions

Q. 1. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$ **is equal to**

(A) $2(\sin x + x\cos\theta) + C$
(B) $2(\sin x - x\cos\theta) + C$
(C) $2(\sin x + 2x\cos\theta) + C$
(D) $2(\sin x - 2x\cos\theta) + C$

Ans. Option (A) is correct.

Explanation: Let,

$$I = \int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$$

$$= \int \frac{(2\cos^2 x - 1 - 2\cos^2\theta + 1)}{\cos x - \cos\theta} dx$$

$$= 2\int \frac{(\cos x + \cos\theta)(\cos x - \cos\theta)}{(\cos x - \cos\theta)} dx$$

$$= 2\int (\cos x + \cos\theta) dx$$

$$= 2\sin x + 2x\cos\theta + C$$

Q. 2. $\int \frac{dx}{e^x + e^{-x}}$ **is equal to**

(A) $\tan^{-1}(e^x) + C$

(B) $\tan^{-1}(e^{-x}) + C$

(C) $\log(e^x - e^{-x}) + C$

(D) $\log(e^x + e^{-x}) + C$

Ans. Option (A) is correct.

Explanation:

Let $I = \int \frac{dx}{e^x + e^{-x}} dx$

$= \int \frac{e^x}{e^{2x}+1} dx$

Also, let $e^x = t$

$\Rightarrow e^x dx = dt$

$\Rightarrow I = \int \frac{dt}{1+t^2}$

$= \tan^{-1} t + C$

$= \tan^{-1}(e^x) + C$

Q. 3. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ **is equal to**

(A) $\frac{-1}{\sin x + \cos x} + C$

(B) $\log|\sin x + \cos x| + C$

(C) $\log|\sin x - \cos x| + C$

(D) $\frac{1}{(\sin x + \cos x)^2}$

Ans. Option (B) is correct.

Explanation:

Let $I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$

$I = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$

$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx$

$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Let $\cos x + \sin x = t$

$\Rightarrow (\cos x - \sin x)\, dx = dt$

$\Rightarrow I = \int \frac{dt}{t}$

$= \log|t| + C$

$= \log|\cos x + \sin x| + C$

Q. 4. $\int \frac{dx}{\sin(x-a)\ \sin(x-b)}$ **is equal to**

(A) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$

(B) $\text{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

(C) $\text{cosec}\ (b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$

(D) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

Ans. Option (C) is correct.

Explanation: Let,

$I = \int \frac{dx}{\sin(x-a)\sin(x-b)}$

$= \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)} dx$

$= \frac{1}{\sin(b-a)} \int \frac{\sin(x-a-x+b)}{\sin(x-a)\sin(x-b)} dx$

$= \frac{1}{\sin(b-a)} \int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)} dx$

$= \frac{1}{\sin(b-a)} \int \frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)} dx$

$= \frac{1}{\sin(b-a)} \int [\cot(x-b) - \cot(x-a)] dx$

$= \frac{1}{\sin(b-a)} [\log|\sin(x-b)| - \log|\sin(x-a)|] + C$

$= \text{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$

Q. 5. $\int \sqrt{1+x^2}\, dx$ **is equal to**

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right| + C$

(B) $\frac{2}{3}(1+x^2)^{3/2} + C$

(C) $\frac{2}{3}x(1+x^2)^{3/2} + C$

(D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left(x+\sqrt{1+x^2}\right) + C$

Ans. Option (A) is correct.

Explanation: It is known that,

$\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

$\therefore \int \sqrt{1+x^2}\, dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right| + C$

Q. 6. $\int \frac{x\,dx}{(x-1)(x-2)}$ **equals**

(A) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

(B) $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

(C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

(D) $\log|(x-1)(x-2)| + C$

Ans. Option (B) is correct.

Explanation:

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1)$ ...(i)

Substituting $x = 1$ and 2 in Eq. (i), we obtain

A = –1 and B = 2

$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$

$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{\frac{-1}{(x-1)} + \frac{2}{(x-2)}\right\} dx$

$= -\log|x-1| + 2\log|x-2| + C$

$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$

Q. 7. $\int \tan^{-1}\sqrt{x}\ dx$ **is equal to**

(A) $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(B) $x\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(C) $\sqrt{x} - x\tan^{-1}\sqrt{x} + C$

(D) $\sqrt{x} - (x+1)\tan^{-1}\sqrt{x} + C$

Ans. Option (A) is correct.

Explanation: Let, $\int 1 \cdot \tan^{-1}\sqrt{x}\ dx$

Apply integration by parts.

$$= x\tan^{-1}\sqrt{x} - \int \frac{x}{2(1+x)\sqrt{x}}dx$$

$$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x}dx$$

Let $\sqrt{x} = t$

$$\frac{dx}{2\sqrt{x}} = dt$$

$$dx = 2tdt$$

$$= x\tan^{-1}\sqrt{x} - \int \frac{t^2}{1+t^2}dt$$

$$= x\tan^{-1}\sqrt{x} - \int dt + \int \frac{1}{1+t^2}dt$$

$$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + c$$

$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + c$$

Q. 8. $\int x^2 e^{x^3} dx$ **is equal to**

(A) $\frac{1}{3}e^{x^3} + C$ (B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$ (D) $\frac{1}{2}e^{x^2} + C$

Ans. Option (A) is correct.

Explanation:

Let $I = \int x^2 e^{x^3} dx$

Also, let $x^3 = t$

$\Rightarrow 3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$

$$= \frac{1}{3}(e^t) + C$$

$$= \frac{1}{3}e^{x^3} + C$$

Q. 9. $\int e^x \sec x(1+\tan x)dx$ **is equal to**

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

Ans. Option (B) is correct.

Explanation: $\int e^x \sec x(1+\tan x)dx$

Let $I = \int e^x \sec x(1+\tan x)dx = \int e^x(\sec x + \sec x\tan x)dx$

Also, let $\sec x = f(x) \Rightarrow \sec x\tan x = f'(x)$

It is known that, $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$

$\therefore I = e^x \sec x + C$

Q. 10. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}dx$ **is equal to**

(A) $\tan x + \cot x + C$ (B) $\tan x + \text{cosec}\, x + C$

(C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

Ans. Option (A) is correct.

Explanation:

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x\cos^2 x}dx = \int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x} - \frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$

$$= \int(\sec^2 x - \text{cosec}^2 x)dx$$

$$= \tan x + \cot x + C$$

Q. 11. $\int \frac{dx}{\sin^2 x\cos^2 x}$ **is equal to**

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

Ans. Option (B) is correct.

Explanation:

Let $I = \int \frac{dx}{\sin^2 x\cos^2 x}$

$$= \int \frac{1}{\sin^2 x\cos^2 x}dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x\cos^2 x}dx + \int \frac{\cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$= \int \sec^2 x dx + \int \text{cosec}^2\, x dx$$

$$= \tan x - \cot x + C$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Find $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx.$

A I R [CBSE OD Set-I, II, III 2020]

Sol.

$$I = \int\left(2(5^{-x}) - \frac{1}{5}(2^{-x})\right)dx$$

$$= -\frac{2}{5^x \log 5} + \frac{1}{5(2^x)\log 2} + C \quad 1$$

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int \frac{2^{x+1} - 5^{x-1}}{10^x}dx = \int \frac{2^{x+1} - 5^{x-1}}{2^x.5^x}dx$$

$$= \int\left(\frac{2^{x+1}}{2^x.5^x} - \frac{5^{x-1}}{2^x.5^x}\right)dx$$

$$= \int \frac{2}{5^x}dx - \int \frac{1}{5 \cdot 2^x}dx$$

$$= 2\int 5^{-x}dx - \frac{1}{5}\int 2^{-x}dx$$

$$= 2\times\frac{-5^{-x}}{\log 5}-\frac{1}{5}\times\frac{-2^{-x}}{\log 2}$$

$$= \frac{1}{5.2^x\log 2}-\frac{2}{5^x\log 5}+C$$

Q. 2. Find : $\int\frac{dx}{\sqrt{x}+x}$ R [CBSE OD Set-I 2020]

Q. 3. Find $\int\sin^5\left(\frac{x}{2}\right).\cos\left(\frac{x}{2}\right)dx$

R&U [CBSE OD Set-II 2020]

Sol. $I = \int\sin^5\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)dx$

Putting $\sin\left(\frac{x}{2}\right) = t$ gives $I = 2\int t^5 dt$ ½

$\therefore$ $I = \frac{t^6}{3}+C = \frac{1}{3}\sin^6\left(\frac{x}{2}\right)+C$ ½

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int\sin^5\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)dx$$

let $\sin\frac{x}{2} = t$

$$\cos\frac{x}{2}\times\frac{1}{2}dx = dt$$

$$\cos\left(\frac{x}{2}\right)dx = 2dt$$

$$I = 2\int t^5 dt$$

$$I = \frac{2t^6}{6}+C$$

$$I = \frac{\sin^6\frac{x}{2}}{3}+C$$

Q. 4. Find $\int e^x(1-\cot x+\text{cosec}^2 x)dx$

R&U [CBSE SQP 2020-21]

Sol. $e^x(1-\cot x)+C$ **1**

[CBSE Marking Scheme 2020]

Detailed Solution:

$e^x(1-\cot x+\text{cosec}^2 x)dx$

$= f(x) = 1-\cot x$

$f'(x) = \text{cosec}\,x$ $\left[\therefore \int e^x(f(x)+f'(x))dx = e^x f(x)+C\right]$

$= e^x(1-\cot x)+C$

Commonly Made Error

- Some students use the formula for integration by parts and make it lengthy.

Answering Tip

- Use the correct formula in the right place.

Q. 5. Find $\int\frac{1}{x(1+x^2)}dx$

A I R&U [CBSE OD Set-III 2020]

Sol. $I = \int\frac{1}{x(1+x^2)}dx = \int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx$ ½

$= \log|x|-\frac{1}{2}(1+x^2)+C$ ½

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int\frac{1}{x(1+x^2)}dx$$

$$\int\frac{x}{x^2(1+x^2)}dx$$

let $1+x^2 = t$

$2xdx = dt$

$xdx = \frac{dt}{2}$

$$= \frac{1}{2}\int\frac{1}{(t-1)t}dt$$

$$\frac{1}{(t-1)(t)} = \frac{A}{t}+\frac{B}{t-1} \quad \text{...(i)}$$

$1 = A(t-1)+Bt$

Put $t = 0$

$A = -1$

Put $t = 1$

$B = 1$

Put $A = -1$ and $B = 1$ ...(ii)

$$\frac{1}{(t-1)t} = \frac{-1}{t}+\frac{1}{t-1}$$

$$\frac{1}{2}\int\frac{1}{(t-1)t}dt = \frac{1}{2}\left[-\int\frac{1}{t}dt+\int\frac{1}{t-1}dt\right]$$

$$= \frac{1}{2}[-\log t+\log|t-1|]+C$$

$$= \frac{1}{2}\left[\log\frac{t-1}{t}\right]+C$$

Put $t = x^2+1$

$$= \frac{1}{2}\left[\log\frac{x^2+1-1}{x^2+1}\right]+C$$

$$= \frac{1}{2}\log\frac{x^2}{1+x^2}+C$$

Concept Applied

Integration by Partial fraction

Q. 6. Find $\int (\cos^2 2x - \sin^2 2x)\,dx$

R [CBSE SQP-2020-21]

Ans : $\int \cos 4x\,dx = \dfrac{\sin 4x}{4} + c$ **1**

[CBSE SQP Marking Scheme 2020]

Commonly Made Error

- Some students use the formula for cos 2A and make the answer complicated.

Answering Tip

- Apply $\cos^2\theta - \sin^2\theta = \cos 2\theta$

Q. 7. Find : $\int \dfrac{3+3\cos x}{x+\sin x}\,dx$.

AI A [CBSE SQP-2020-21]

Sol. Let $x + \sin x = t$

So, $(1 + \cos x)dx = dt$

$I = 3\int \dfrac{dt}{t} = 3\log|t| + c = 3\log|(x+\sin x)| + c$

or directly by writing formula

$\int \dfrac{f'(x)}{f(x)}dx = \log|f(x)| + c$ **1**

[CBSE SQP Marking Scheme 2020]

Commonly Made Error

- Some students split the numerator and apply by parts which is too lengthy.

Answering Tip

- Put the denominator as t; if numerator is the derivative of denominator.

Q. 8. Find $\int xe^{(1+x^2)}dx$.

R [CBSE SQP-2020-21]

Sol. Let $(1 + x^2) = t$

so $2x\,dx = dt$

$\Rightarrow I = \dfrac{1}{2}\int e^t\,dt = \dfrac{1}{2}e^t + c = \dfrac{1}{2}e^{(1+x^2)} + c$ **1**

[CBSE SQP Marking Scheme, 2020]

Q. 9. Find : $\int \dfrac{\sin^2 x - \cos^2 x}{\sin x \cos x}\,dx$ R&U [O.D. Set I 2017]

Q. 10. Find : $\int \dfrac{1}{x(1+\log x)}\,dx$

R&U [Delhi Comptt. 2017] [NCERT]

Sol. Let $1 + \log x = t$

$\left(\dfrac{1}{x}\right)dx = dt$

$dx = x\,dt$

$I = \int \dfrac{1}{t}dt$ ½

$= \int \dfrac{1}{t}dt$

$= \log|t| + C$

$= \log|1 + \log x| + C$ ½

[CBSE Marking Scheme 2017]

Q. 11. Find : $\int \dfrac{3x}{3x-1}\,dx$ R&U [O.D. Comptt. 2017]

Q. 12. Find : $\int \dfrac{dx}{\sqrt{9-4x^2}}$ [CBSE Foreign Set II 2020]

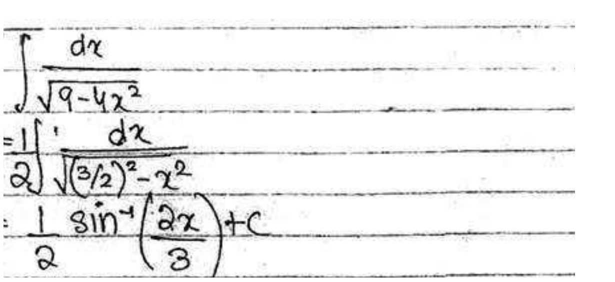
Sol. $\int \dfrac{dx}{\sqrt{9-4x^2}} = \int \dfrac{dx}{\sqrt{(3)^2-(2x)^2}}$

$= \dfrac{1}{2}\sin^{-1}\left(\dfrac{2x}{3}\right) + C$ **1**

[CBSE Marking Scheme 2020]

OR

Topper Answer, 2020



Q. 13. Find : $\int \dfrac{2x}{\sqrt[3]{x^2+1}}\,dx$ [CBSE Foreign Set II 2020]

Sol. Let, $x^2 + 1 = t$ $\therefore 2x\,dx = dt$

$\int \dfrac{2x}{\sqrt[3]{x^2+1}}dx = \int \dfrac{1}{\sqrt[3]{t}}dt$ ½

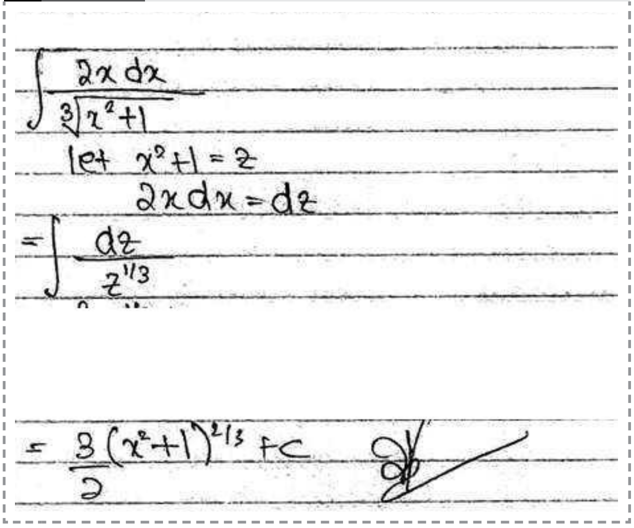
$\int t^{-\frac{1}{3}}dt = \dfrac{3}{2}t^{\frac{2}{3}} + c$

$$= \frac{3}{2}(x^2+1)^{\frac{2}{3}} + c \qquad ½$$

[CBSE Marking Scheme 2020]

OR

Topper Answer, 2020


$\int \frac{2x\,dx}{\sqrt[3]{x^2+1}}$

let $x^2+1 = z$

$2x\,dx = dz$

$= \int \frac{dz}{z^{1/3}}$

$= \frac{3}{2}(x^2+1)^{2/3} + c$

Short Answer Type Questions-I (2 marks each)

Q. 1. Find $\int \frac{\log x}{(1+\log x)^2} dx$. **[SQP 2021-22]**

Sol. $\int \frac{\log x}{(1+\log x)^2} dx$

$$= \int \frac{\log x + 1 - 1}{(1+\log x)^2} dx$$

$$= \int \frac{1}{1+\log x} dx - \int \frac{1}{(1+\log x)^2} dx \qquad ½$$

$$= \frac{1}{1+\log x} \times x - \int \frac{-1}{(1+\log x)^2} \times \frac{1}{x} \times x\,dx$$

$$- \int \frac{1}{(1+\log x)^2} dx \qquad 1$$

$$= \frac{x}{1+\log x} + c \qquad ½$$

[CBSE Marking Scheme 2022]

Q. 2. Find $\int \frac{\sin 2x}{\sqrt{9-\cos^4 x}} dx$. **[SQP 2021]**

Sol. $I = \int \frac{\sin 2x}{\sqrt{9-\cos^4 x}} dx$

Put $\cos^2 x = t$

$\Rightarrow -2\cos x \sin x\,dx = dt$

$\Rightarrow \sin 2x\,dx = -dt \qquad 1$

The given integral

$$I = -\int \frac{dt}{\sqrt{3^2 - t^2}}$$

$$= -\sin^{-1}\frac{t}{3} + c$$

$$= -\sin^{-1}\frac{\cos^2 x}{3} + c \qquad 1$$

[CBSE Marking Scheme 2022]

Q. 3. Find $\int \frac{1}{\cos^2 x(1-\tan x)^2} dx$

[CBSE SQP 2020-21]

Concept Applied

Integration by Substitution

Q. 4. Find $\int \frac{x}{x^2+3x+2} dx$.

[CBSE Delhi Set-I 2020]

Sol. $\int \frac{x}{x^2+3x+2} dx = \int \frac{x}{(x+1)(x+2)} dx$

$$= \int \left(\frac{-1}{x+1} + \frac{2}{x+2}\right) dx \qquad 1$$

$$= -\log|x+1| + 2\log|x+2| + C \qquad 1$$

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int \frac{x\,dx}{x^2+3x+2} = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2} - \frac{3}{2}\int \frac{dx}{x^2+3x+2}$$

$$= \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2} \times \frac{1}{2\times\frac{1}{2}} \log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + C$$

$$\Rightarrow \int \frac{x\,dx}{x^2+3x+2} = \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$$

Q. 5. Find $\int \frac{x+1}{x(1-2x)} dx$.

[CBSE Delhi Set-II 2020]

Sol. $\int \frac{x+1}{x(1-2x)}dx = \int\left(\frac{1}{x}+\frac{3}{1-2x}\right)dx$ **1**

$= \log|x| - \frac{3}{2}\log|1-2x| + c$ **1**

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int \frac{x+1}{x(1-2x)}dx = \int \frac{x}{x(1-2x)}dx + \int \frac{1}{x(1-2x)}dx$$

$$\Rightarrow \int \frac{x+1}{x(1-2x)}dx = \int \frac{1}{(1-2x)}dx + \int \frac{1}{x}dx + \int \frac{2}{(1-2x)}dx$$

$$\Rightarrow \int \frac{x+1}{x(1-2x)}dx = -\frac{1}{2}\log|1-2x| + \log|x| - 2\times\frac{1}{2}\log|1-2x| + C$$

$$\Rightarrow \int \frac{x+1}{x(1-2x)}dx = -\frac{3}{2}\log|1-2x| + \log|x| + C$$

Q. 6. Evaluate $\int \frac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}\,dx$.

A I R [CBSE Delhi Set-II 2020]

Sol. $\int \frac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}dx = \frac{1}{2}\int t\,dt,$

where $\sin^{-1}(x^2) = t$ **1**

$= \frac{t^2}{4} + C$ **½**

$= \frac{1}{4}(\sin^{-1}x^2)^2 + C$ **½**

[CBSE Marking Scheme 2020]

Detailed Solution:

Given that $\int \frac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}dx$

Put $\sin^{-1}(x^2) = y$

$\Rightarrow \frac{x}{\sqrt{1-x^4}}dx = \frac{dy}{2}$

$\Rightarrow \frac{1}{2}\int y\,dy = \frac{1}{4}\times y^2 + C$

$\therefore \int \frac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}dx = \frac{1}{4}(\sin^{-1}(x^2))^2 + C$

Q. 7. Find $\int \frac{x+1}{(x+2)(x+3)}dx$.

A I R [CBSE Delhi Set-III 2020]

Sol. $\int \frac{x+1}{(x+2)(x+3)}dx = \int\left(-\frac{1}{x+2}+\frac{2}{x+3}\right)dx$ **1**

$= -\log|x+2| + 2\log|x+3| + C$ **1**

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int \frac{x+1}{(x+2)(x+3)}dx = \int \frac{x+2-1}{(x+2)(x+3)}dx$$

$$\Rightarrow \int \frac{x+1}{(x+2)(x+3)}dx = \int \frac{1}{(x+3)}dx - \int\left\{\frac{1}{(x+2)} - \frac{1}{(x+3)}\right\}dx$$

$= \log|x+3) - \log|x+2| + \log|x+3| + C$

$$\Rightarrow \int \frac{x+1}{(x+2)(x+3)}dx = 2\log|x+3| - \log|x+2| + C$$

Q. 8. Find : $\int \frac{e^x(x-3)}{(x-1)^3}dx$.

R&U [CBSE OD-1, 2019, SQP 2018-19]

Sol. $\int \frac{e^x(x-3)}{(x-1)^3}dx = \int e^x\left[\frac{(x-1)-2}{(x-1)^3}\right]dx$ **1**

$= \int e^x\left[\frac{1}{(x-1)^2} + \left(\frac{-2}{(x-1)^3}\right)\right]dx$ **½**

$\left(\because \int e^x[f(x)+f'(x)]dx = e^x f(x) + c\right)$

$f(x) = \frac{1}{(x-1)^2}$

$= \frac{e^x}{(x-1)^2} + C$ **½**

[CBSE Marking Scheme 2019]

Commonly Made Error

- Some candidates use By parts in $\int e^x\,|f(x)+f'(x)|$ in both parts which is wrong.

Answering Tip

- $\int e^x\{f(x)+f'(x)\}$ is specific situation in integration by parts. This needs to be sufficiently practiced by students.

Q. 9. Find : $\int \sin^{-1}(2x)\,dx$. R [Delhi Set-I-2019]

Q. 10. Find : $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x\cos^2 x}dx$

A I R [CBSE OD Set I-2019] [NCERT]

Sol. $\int \frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}dx = \int(\sec x.\tan x + \text{cosec}\,x.\cot x)dx$ **1**

$= \sec x - \text{cosec}\,x + c$ **1**

[CBSE Marking Scheme, 2019]

Detailed Solution :

$$\text{Let } I = \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \left(\frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} \right) dx$$

$$= \int (\tan x.\sec x + \cot x.\text{cosec}\, x) dx$$

$$= \int \tan x.\sec x\, dx + \int \cot x.\text{cosec}\, x dx$$

$$= \sec x - \text{cosec}\, x + c$$

Q. 11. Find : $\int \sqrt{3-2x-x^2}\, dx$ R [CBSE OD Set-I, 2019]

Sol. $\int \sqrt{3-2x-x^2}\, dx = \int \sqrt{2^2-(x+1)^2}\, dx$ **1**

$$= \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + c$$ **1**

[CBSE Marking Scheme, 2019]

Detailed Solution :

$$\text{Let } I = \int \sqrt{3-2x-x^2}\, dx$$

$$= \int \sqrt{3-(2x+x^2)}\, dx$$

$$= \int \sqrt{3-(x^2+2x+1-1)} dx$$

$$= \int \sqrt{3-(x^2+2x+1)+1}\, dx$$

$$= \int \sqrt{4-(x+1)^2}\, dx$$

$$= \int \sqrt{2^2-(x+1)^2}\, dx$$

$$= \frac{1}{2}(x+1)\sqrt{2^2-(x+1)^2} + \frac{2^2}{2}\sin^{-1}\frac{(x+1)}{2} + c$$

$$\left[\text{as } \int \sqrt{a^2-x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{2}(x+1)\sqrt{4-(x^2+1+2x)} + 2\sin^{-1}\frac{(x+1)}{2} + c$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\sin^{-1}\frac{(x+1)}{2} + c$$

Q. 12. Find : $\int \sin x \log \cos x\, dx$

R [CBSE Delhi Set-III 2019]

Sol. $I = \int \sin x.\log(\cos x) dx$

$\cos x = t \Rightarrow I = -\int \log t.1 dt$ **1**

$= -\left[t.\log t - \int \frac{1}{t}.t dt\right]$ ½

$= t(1-\log t) + c = \cos x(1-\log(\cos x)) + c$ ½

[CBSE Marking Scheme, 2019]

Detailed Solution :

Let, $I = \int \sin x.\log \cos x\, dx$

Put $\cos x = t$

$\Rightarrow$ $-\sin x\, dx = dt$

$\therefore$ $I = -\int \log t\, dt$

$= -\int 1 \log t\, dt$

$= -\left[(t\log t) - \left\{\int t\, \frac{1}{t} dt\right\}\right]$

$= -\left[t\log t - \int 1\, dt\right]$

$= -t\log t + t + c$

$= t(1-\log t) + c$

On substituting value of t, we get

$I = \cos x\, [1 - \log(\cos x)] + c$

AI **Q. 13. Find :** $\int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x} dx$.

U [CBSE Delhi Set-III 2019]

Sol. $$I = \int \frac{\tan^2 x.\sec^2 x}{1-(\tan^3 x)^2} dx$$

$\text{Put } \tan^3 x = t \Rightarrow I = \frac{1}{3}\int \frac{dt}{1-t^2}$ **1**

$= \frac{1}{6}\log\left|\frac{1+t}{1-t}\right| + c = \frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right| + c$

½ + ½

[CBSE Marking Scheme, 2019]

OR

Topper Answer, 2019

$$I = \int \frac{\tan^2 x \cdot \sec^2 x}{1-\tan^6 x} dx$$

$$I = \int t^2\,dt - \int t^2\,dt$$

~~$3t^2dt$~~

$$I = \frac{1}{3}\int \frac{dt}{(1)^2 - t^2}$$

$$= \frac{1}{3}\cdot\frac{1}{2.(1)}\log\left|\frac{1+t}{1-t}\right| + C \quad \left[\because \int \frac{dx}{a^2-x^2} = \log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$I = \frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right| + c$$

Commonly Made Error

- Some students put $\tan x = t$ and make it more complicated.

Answering Tips

- Practice more problems to identify the right substitution.

Q. 14. Find : $\int \sqrt{1-\sin 2x}\ dx,\ \frac{\pi}{4} < x < \frac{\pi}{2}$

U [CBSE Delhi Set-I 2019]

Q. 15. Find : $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}dx.$

A I R [CBSE Delhi Set-I 2019] [NCERT]

Sol. Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ ½

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}dx = \int \frac{dt}{\sqrt{t^2 + 4}}$$ ½

$$= \log\left|t + \sqrt{t^2 + 4}\right| + c$$ ½

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + c$$ ½

[CBSE Marking Scheme, 2019]

Detailed Solution :

$$\text{Let I} = \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$I = \int \frac{dt}{\sqrt{2^2 + t^2}}$$

$$= \log\left|t + \sqrt{2^2 + t^2}\right| + c$$

$$= \log\left|\tan x + \sqrt{4 + \tan^2 x}\right| + c$$

Q. 16. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$.

A I R [CBSE OD-2018] [NCERT]

Sol. $I = \int \frac{1 - 2\sin^2 x + 2\sin^2 x}{\cos^2 x}dx$ ½

$= \int \sec^2 x\, dx$ ½

$= \tan x + C$ **1**

[CBSE Marking Scheme 2018]

OR

Topper Answer, 2018

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$$

$$\int \frac{1 - 2\sin^2 x + 2\sin^2 x}{\cos^2 x}dx$$

$$\int \frac{dx}{\cos^2 x}$$

$$\int \sec^2 x\, dx$$

$\Rightarrow \tan x + c$ Ans.

Q. 16. Find : $\int \frac{\cos x\, dx}{\sqrt{8 - \sin^2 x}}$ R&U [O.D. Comptt. 2017]

Short Answer Type Questions-II (3 marks each)

Q. 1. Find : $\int \frac{x+1}{(x^2+1)x}dx.$

R&U [CBSE SQP 2021-2022]

Sol. Let

$$\frac{x+1}{(x^2+1)x} = \frac{Ax+B}{x^2+1}+\frac{C}{x} = \frac{(Ax+B)x+C(x^2+1)}{(x^2+1)x}$$ ½

$\Rightarrow \quad x+1 = (Ax+B)x+C(x^2+1)$(An identity)

Equating the coefficients, we get

$B = 1, C = 1, A + C = 0$

Hence, $A = -1, B = 1, C = 1$ ½

The given integral

$$\int \frac{x+1}{(x^2+1)x} = \int \frac{-x+1}{x^2+1}dx + \int \frac{1}{x}dx$$

$$= \frac{-1}{2}\int \frac{2x-2}{x^2+1}dx + \int \frac{1}{x}dx$$ ½

$$= \frac{-1}{2}\int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx + \int \frac{1}{x}dx$$

$$= \frac{-1}{2}\log(x^2+1)+\tan^{-1}x$$

$$+\log|x|+c$$ 1½

[CBSE Marking Scheme 2022]

Q. 2. Find $\int \frac{x^2+1}{(x^2+2)(x^2+3)}dx.$ R&U [CBSE SQP 2020-21]

Sol. Put $x^2 = y$ to make partial fractions ½

$$\frac{x^2+1}{(x^2+2)(x^2+3)} = \frac{y+1}{(y+2)(y+3)}$$

Let $\frac{y+1}{(y+2)(y+3)} = \frac{A}{y+2}+\frac{B}{y+3}$

$\Rightarrow \quad y + 1 = A(y + 3) + B(y + 2)$...(1) ½

Comparing coefficients of y and constant terms on both sides of (1) we get

$A + B = 1$

and $3A + 2B = 1$

Solving, we get

$A = -1, B = 2$ 1

$$\int \frac{x^2+1}{(x^2+2)(x^2+3)}dx = \int \frac{-1}{x^2+2}dx + 2\int \frac{1}{x^2+3}dx$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$$ 1

$$+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$$

[CBSE SQP Marking Scheme 2020-21]

Commonly Made Error

- Students apply the formula for quadratic factors in partial fractions.

Answering Tip

- Learn and practice more problems on partial fractions.

Q. 3. Find : $\int \frac{3x+5}{x^2+3x-18}dx$

R&U [CBSE Delhi Set I-2019]

Q. 4. Find : $\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx.$

A [CBSE OD Set-I, 2019] [NCERT] [Delhi 2016] [OD 2016] [OD 2015]

Sol. $\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{A}{x+2}+\frac{Bx+C}{x^2+1}$

or $A = \frac{3}{5}, B=\frac{2}{5}, C=\frac{1}{5}$ 1

$\therefore \quad I = \frac{3}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{2x}{x^2+1}dx+\frac{1}{5}\int\frac{dx}{x^2+1}$ 1

or $I = \frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$ 1

[CBSE Marking Scheme 2019] (Modified)

Detailed Solution :

Let $I = \int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$

$= \int \frac{x^2+x+1}{(x+2)(x^2+1)}dx$

Let $\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2}+\frac{Bx+C}{x^2+1}$

or $x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 2)$

Put, $x = -2$

We get

$(-2)^2 + (-2) + 1 = A[(-2)^2 + 1]$

or $4-2+1 = 5A$

or $A = \frac{3}{5}$

Let $x = -1$, we get

$(-1)^2 + (-1) + 1 = A(1 + 1) + (-B + C)(-1 + 2)$

or $1 = 2A - B + C$

or $B-C = \frac{6}{5}-1=\frac{1}{5}$

$\therefore \quad B-C = \frac{1}{5}$ ...(i)

Put $x = 1$, we get

or $1 + 1 + 1 = A(1 + 1) + (B + C)(1 + 2)$

or $3 = 2A + 3B + 3C$

or $3B + 3C = 3-\frac{6}{5}=\frac{9}{5}$

or $\qquad B + C = \frac{3}{5}$ ...(ii)

Solving eqn. (i) and eqn. (ii), we get

$$B = \frac{2}{5} \text{ and } C = \frac{1}{5}$$

$$\therefore \quad \frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{\frac{3}{5}}{(x+2)} + \frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1}$$

$$= \frac{3}{5}\left(\frac{1}{x+2}\right) + \frac{1}{5}\left(\frac{2x+1}{x^2+1}\right)$$

$$\therefore \quad I = \frac{3}{5}\int\frac{dx}{x+2} + \frac{1}{5}\int\frac{2xdx}{x^2+1} + \frac{1}{5}\int\frac{dx}{x^2+1}$$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log(x^2+1) + \frac{1}{5}\tan^{-1}x + C$$

Commonly Made Error

- In most of the cases, candidates who use appropriate substitution do not give the final answer in terms of x but leave in terms of the new variable.

Answering Tip

- Practice needs to be done in integration using partial fraction.

Q. 5. Find : $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$.

A I U **[CBSE Delhi Set-III-2019]**

Sol. $I = \int\frac{\cos x}{(1+\sin x)(2+\sin x)}dx.$ Put $\sin x = t$ ½

$$= \int\frac{dt}{(1+t)(2+t)} = \int\left(\frac{1}{1+t} - \frac{1}{2+t}\right)dt \qquad 2$$

$$= \log\left|\frac{1+t}{2+t}\right| + c = \log\left|\frac{1+\sin x}{2+\sin x}\right| + c \qquad ½$$

[CBSE Marking Scheme, 2019] (Modified)

Detailed Solution :

$$\text{Let I} = \int\frac{\cos x}{(1+\sin x)(2+\sin x)}dx$$

Let $\qquad 2 + \sin x = t$

$\Rightarrow \qquad \cos x\, dx = dt$

Also, $1 + (1 + \sin x) = t$

$\Rightarrow \qquad 1 + \sin x = t - 1$

Now, $\qquad I = \int\frac{dt}{(t-1)t}$

$$= \int\frac{1}{t-1}dt - \int\frac{1}{t}dt$$

$$= \log(t-1) - \log t + c$$

$$I = \log(2 + \sin x - 1) - \log(2 + \sin x) + c$$

Hence, $\int\frac{\cos x}{(1+\sin x)(2+\sin x)}dx = \log\left(\frac{1+\sin x}{2+\sin x}\right) + c$

Commonly Made Error

- Many candidates apply incorrect substitution and are not able to reduce in partial fraction.

Answering Tip

- If the denominator has the derivative in numerator then put as t and reduce into partial fraction.

Q. 6. Find : $\int\frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

U **[CBSE OD-2018]**

Sol. Put $\sin x = t \Rightarrow \cos x\, dx = dt$

Put $I = \int\frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

$$= \int\frac{2}{(1-t)(1+t^2)}dt \qquad ½$$

Let $\frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$ ½

By solving we get

$A = 1, B = 1, C = 1$

$$\therefore \quad I = \int\frac{1}{1-t}dt + \frac{1}{2}\int\frac{2t}{1+t^2} + \int\frac{1}{1+t^2}dt \qquad 1½$$

$$= -\log|1-t| + \frac{1}{2}\log|1+t^2| + \tan^{-1}t + C$$

$$= -\log(1-\sin x) + \frac{1}{2}\log(1+\sin^2 x) + \tan^{-1}(\sin x) + C \qquad ½$$

[CBSE Marking Scheme 2018] (Modified)

OR

Topper Answer, 2018

Put $\sin x = t$

$\cos x\, dx = dt$

$$2\int \frac{dt}{(1-t)(1+t^2)}$$

Now by partial fraction,

$$\frac{1}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$$

$$1 = A(1+t^2) + (Bt+C)(1-t)$$

$$1 = A + At^2 + Bt - Bt^2 + C - Ct$$

$$A + C = 1 \quad \text{—(i)}$$

$$A - B = 0 \quad \text{—(ii)}$$

$$B - C = 0 \quad \text{—(iii)}$$

(i) + (iii)

$$A + C + B - C = 1$$

$$A + B = 1 \quad \text{—(iv)}$$

$$A - B = 0 \quad \text{—(ii)}$$

$$2A = 1$$

$$A = 1/2, \quad B = 1/2, \quad C = 1/2$$

$$2\int \frac{1/2}{1-t} + \frac{1/2\, t + 1/2}{1+t^2}\, dt$$

$$2 \times \frac{1}{2} \int \frac{dt}{1-t} + 2 \times \frac{1}{2} \int \frac{t+1}{t^2+1}\, dt$$

$$\int \frac{dt}{1-t} + \int \frac{t\, dt}{t^2+1} + \int \frac{dt}{t^2+1}$$

$$\int \frac{dt}{1-t} + \frac{1}{2}\int \frac{2t\, dt}{t^2+1} + \int \frac{dt}{t^2+1}$$

$$-\log(1-t) + \frac{1}{2}\log(t^2+1) + \tan^{-1} t + C.$$

$$-\log(1-\sin x) + \frac{1}{2}\log(1+\sin^2 x) + \tan^{-1}(\sin x) + C.$$

$$\Rightarrow \frac{1}{2}\log(1+\sin^2 x) - \log(1-\sin x) + \tan^{-1}(\sin x) + C \quad \text{Ans}$$

$$= \log(1+\sin^2 x)^{1/2} - \log(1-\sin x) + \tan^{-1}(\sin x) + C.$$

$$= \log\left|\frac{\sqrt{1+\sin^2 x}}{1-\sin x}\right| + \tan^{-1}(\sin x) + C.$$

Q. 7. Find : $\displaystyle\int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}\, dx$ R&U [Delhi 2017]

Sol. $\displaystyle\int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}\, dx$

$$= \int \frac{(3\sin x - 2)\cos x}{\sin^2 x - 7\sin x + 12}\, dx \quad ½$$

put $\sin x = y$, or $\cos x\, dx = dy$

$$= \int \frac{(3y-2)dy}{y^2 - 7y + 12} \quad ½$$

$$= \int \frac{(3y-2)dy}{(y-4)(y-3)}$$

Let, $\displaystyle\frac{3y-2}{(y-4)(y-3)} = \frac{A}{y-4} + \frac{B}{y-3}$

$3y - 2 = A(y - 3) + B(y - 4)$ ½

Put $y = 3$ $\quad 9 - 2 = A \times 0 + B(3 - 4)$

$-7 = B$

$y = 4$ $\quad 12 - 2 = A(4 - 3) + B \times 0$

$A = 10$

$$= \int \left(\frac{10}{y-4} - \frac{7}{y-3}\right) dy$$

$= 10 \log|y - 4| - 7 \log|y - 3| + C$ **1**

$= 10 \log|\sin x - 4| - 7 \log|\sin x - 3| + C$ ½

[CBSE Marking Scheme 2017] (Modified)

Q. 8. Find : $\int \frac{2x}{(x^2+1)(x^4+4)}dx$ R&U [Delhi 2017]

Sol. $\int \frac{2x}{(x^2+1)(x^4+4)}dx$

$= \int \frac{dy}{(y+1)(y^2+4)}$

[put $x^2 = y$ or $2x\,dx = dy$]

$= \frac{1}{(y+1)(y^2+4)} = \frac{A}{y+1} + \frac{By+C}{y^2+4}$

$1 = A(y^2+4) + (By+C)(y+1)$

Put $y = -1$ $\quad 1 = A(1+4) + (-B+C)(0)$

$\therefore \quad A = \frac{1}{5}$

Put $y = 0$ $\quad 1 = 4A + C$

$1 = \frac{4}{5} + C$

$\therefore \quad C = 1 - \frac{4}{5} = \frac{1}{5}$

Put $y = 1$ $\quad 1 = 5A + 2B + 2C$

$1 = 1 + 2B + \frac{2}{5}$

$B = \frac{-1}{5}$

$A = \frac{1}{5}, B = \frac{-1}{5}, C = \frac{1}{5}$ **1**

$\frac{1}{(y+1)(y^2+4)} = \frac{1}{5(y+1)} + \frac{\frac{1}{5} - \frac{1}{5}y}{y^2+4}$ ½

$\int \frac{dy}{(y+1)(y^2+4)} = \frac{1}{5}\log|y+1| + \frac{1}{10}\tan^{-1}\frac{y}{2}$

$-\frac{1}{10}\log(y^2+4) + C$ **1**

$= \frac{1}{5}\log(x^2+1) + \frac{1}{10}\tan^{-1}\frac{x^2}{2}$

$-\frac{1}{10}\log(x^4+4) + C$ ½

[CBSE Marking Scheme 2017] (Modified)

Q. 9. Find : $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

R&U [O.D. Set I, 2017]

Sol. $I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$ ½

$= \int \frac{dt}{(4+t^2)(1+4t^2)}$, where $\sin\theta = t$ ½

$= \int \frac{-\frac{1}{15}}{4+t^2}dt + \int \frac{\frac{4}{15}}{1+4t^2}dt$ ½

$= -\frac{1}{30}\tan^{-1}\left(\frac{t}{2}\right) + \frac{4}{30}\tan^{-1}(2t) + c$ **1**

$= -\frac{1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right) + \frac{2}{15}\tan^{-1}(2\sin\theta) + c$ ½

[CBSE Marking Scheme 2017] (Modified)

Q. 10. Find : $\int \frac{x}{(x^2+1)(x-1)}dx.$ A I R&U [NCERT]

[O.D. Set I, II, III Comptt. 2015]
[Delhi Comptt. 2017]

Q. 11. Find : $\int \frac{(2x-5)e^{2x}}{(2x-3)^3}dx.$ R&U [CBSE 2016]

Sol. Let $\quad 2x = t$

$I = \frac{1}{2}\int \frac{(t-5)}{(t-3)^3}e^t dt$ **1**

$= \frac{1}{2}\int \left[\frac{1}{(t-3)^2} - \frac{2}{(t-3)^3}\right]e^t dt$ **1**

$\int e^x f(x) + f'(x) = e^x f(x) + C$

$= \frac{1}{2}\frac{1}{(t-3)^2}e^t + C$

$= \frac{1}{2}\frac{1}{(2x-3)^2}e^{2x} + C$ **1**

[CBSE Marking Scheme 2016]

Commonly Made Error

- Several candidates commit the mistake of ignoring the constant of integration and use of partial fraction.

Answering Tip

- Reduce the given expression in proper fraction then apply partial fraction.

Long Answer Type Questions-I (4 marks each)

Q. 1. Find : $\int x \sin^{-1} x \, dx$.

A I A [Foreign 2016] [NCERT]
[Delhi Set I, II, III Comptt. 2016]

Sol. $\int x\sin^{-1} x\,dx = \sin^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$ **1**

$= \frac{x^2 \cdot \sin^{-1} x}{2} + \frac{1}{2}\left\{\int \sqrt{1-x^2}\,dx - \int \frac{1}{\sqrt{1-x^2}}dx\right\}$

$= \frac{x^2 \cdot \sin^{-1} x}{2} + \frac{1}{2}\left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x - \sin^{-1} x\right\}$ **1**

or $\frac{x^2 \cdot \sin^{-1} x}{2} + \frac{x\sqrt{1-x^2}}{4} - \frac{1}{4}\cdot\sin^{-1} x + C$ **1**

[CBSE Marking Scheme 2016] (Modified)

Q. 2. Find : $\int \frac{\sin x - x\cos x}{x(x+\sin x)}dx$.

A [Delhi Set I, II, III Comptt. 2016]

Sol. $\int \frac{\sin x - x\cos x}{x(x+\sin x)}dx = \int \frac{(x+\sin x)-(x+x\cos x)}{x(x+\sin x)}dx$ **1**

$= \int\left\{\frac{1}{x} - \frac{1+\cos x}{x+\sin x}\right\}dx$ **1**

$= \log|x| - \log|x+\sin x| + C$

$= \log\left|\frac{x}{x+\sin x}\right| + C$ **1**

[CBSE Marking Scheme 2016] (Modified)

Q. 3. Find : $\int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx$

A I R&U [O.D. Set II, 2016] [NCERT]

Topper Answer, 2016

Let $I = \int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx$

$= \int \log(\log x)\,dx + \int \frac{dx}{(\log x)^2}$

$= I_1 + I_2 + C$ (Say) $\{C$ = Arbitrary constant$\}$

$I_1 = \int \log(\log x)\,dx$ & $I_2 = \int \frac{dx}{(\log x)^2}$

Consider $I_1 = \int \log(\log x)\cdot 1\,dx$

$= x\log(\log x) - \int x \times \frac{1}{\log x} \times \frac{1}{x}dx$ {Applying Integration by parts}

$= x\log(\log x) - \int \frac{dx}{\log x}\times 1$

$= x\log(\log x) - \left[\frac{x}{\log x} - \int x \times \frac{-1}{(\log x)^2} \times \frac{1}{x}dx\right]$ {Applying Integration by parts}

$I_1 = x\log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2}$

But $\int \frac{dx}{(\log x)^2} = I_2$

$\therefore I = I_1 + I_2 + C = x\log(\log x) - \frac{x}{\log x} + C$

Q. 4. Find : $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}dx.$ R&U [Delhi 2016]

Sol. Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}dx$

$$= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}\, dx$$

Put $x^{3/2} = a^{3/2} t$

or $\frac{3}{2} x^{1/2} dx = a^{3/2} dt$

or $\sqrt{x}\; dx = \frac{2}{3} a^{3/2} dt$ **1**

$$\therefore \quad I = \int \frac{\frac{2}{3} a^{3/2}}{\sqrt{(a^{3/2})^2 - (a^{3/2} t)^2}}\, dt$$

$$= \frac{2}{3} a^{3/2} \int \frac{dt}{a^{3/2} \sqrt{1 - t^2}} \quad \textbf{1}$$

$$= \frac{2}{3} \int \frac{dt}{\sqrt{1 - t^2}}$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{t}{1}\right) + C \quad \textbf{1}$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C \quad \left[\text{Put } t = \frac{x^{3/2}}{a^{3/2}}\right]$$

$$= \frac{2}{3} \sin^{-1}\left(\sqrt{\frac{x^3}{a^3}}\right) + C \quad \textbf{1}$$

Q. 5. Evaluate the following indefinite integral :

$$\int \frac{\sin\phi}{\sqrt{\sin^2\phi + 2\cos\phi + 3}}\, d\phi .$$

A **[S.Q.P. 2015-16]**

Sol. Let

$$I = \int \frac{\sin\phi}{\sqrt{\sin^2\phi + 2\cos\phi + 3}}\, d\phi$$

$$= \int \frac{\sin\phi}{\sqrt{1 - \cos^2\phi + 2\cos\phi + 3}}\, d\phi \quad ½$$

$$= \int \frac{\sin\phi}{\sqrt{-\cos^2\phi + 2\cos\phi + 4}}\, d\phi$$

put $\cos\phi = t$

$-\sin\phi\, d\phi = dt$

$\sin\phi\, d\phi = -dt$ **1**

$$= \int \frac{-1}{\sqrt{-t^2 + 2t + 4}}\, dt \quad ½$$

$$= -\int \frac{1}{\sqrt{(\sqrt{5})^2 - (t-1)^2}}\, dt \quad \textbf{1}$$

$$= -\sin^{-1}\frac{t-1}{\sqrt{5}} + C \quad ½$$

$$= -\sin^{-1}\frac{\cos\phi - 1}{\sqrt{5}} + C \quad ½$$

Q. 6. Find : $\int \frac{\sec x}{1 + \operatorname{cosec} x}\, dx$ R&U **[S.Q.P. 2017-18]**

Sol. $\int \frac{\sec x}{1 + \operatorname{cosec} x}\, dx$

$$= \int \frac{\sin x}{\cos x (1 + \sin x)}\, dx$$

$$= \int \frac{\sin x \cos x}{(1 + \sin x)^2 (1 - \sin x)}\, dx \quad \textbf{1}$$

$$= \int \frac{t}{(1+t)^2 (1-t)}\, dt$$

[$\sin x = t$ or $\cos x\, dx = dt$]

Let, $\frac{t}{(1+t)^2(1-t)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{C}{1-t}$ **1**

or $t = A(1 + t)(1 - t) + B(1 - t) + C(1 + t)^2$

(an identity)

Put $t = -1, -1 = -2B$, *i.e.*, $B = -½$. Put, $t = 1$,

$1 = 4C$, *i.e.*, $C = \frac{3}{4}$. Put $t = 0$,

$0 = A + B + C$, which gives $A = \frac{3}{4}$ **1**

Therefore the required integral

$$= \frac{3}{4} \int \frac{1}{1+t}\, dt + \frac{-1}{2} \int \frac{1}{(1+t)^2}\, dt + \frac{1}{4} \int \frac{1}{(1-t)}\, dt$$

$$= \frac{3}{4} \log|1+t| + \frac{-1}{2} \times \frac{-1}{1+t} - \frac{1}{4} \log|1-t| + c$$

$$= \frac{3}{4} \log|1 + \sin x| - \frac{1}{2} \times \frac{1}{1 + \sin x}$$

$$- \frac{1}{4} \log|1 - \sin x| + c \quad \textbf{1}$$

$$= \frac{1}{4} \log \frac{|1 + \sin x|^3}{|1 - \sin x|} - \frac{1}{2} . \frac{1}{1 + \sin x} + c$$

Q. 7. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)}\, dx$. **[Foreign 2015]**

Sol. $\int \frac{\sin(x-a)}{\sin(x+a)}\, dx = I$

Let, $x + a = t$ or $dx = dt$

$x = (t - a)$

$$= \int \frac{\sin(t - a - a)}{\sin t}\, dt \quad (\because x = t - a) \quad \textbf{1}$$

$$= \int \frac{\sin(t - 2a)}{\sin t}\, dt = \int \frac{\sin t . \cos 2a - \cos t . \sin 2a}{\sin t}\, dt \quad ½$$

$$= \int [\cos 2a - \cot t . \sin 2a]\, dt \quad \textbf{1}$$

$$= \cos 2a \int dt - \sin 2a \int \cot t\, dt \quad ½$$

$= t . \cos 2a - \sin 2a \log |\sin t| + C$

$= (x + a) \cos 2a - \sin 2a \log |\sin (x + a)| + C$ **1**

Long Answer Type Questions-II (5 marks each)

Q. 1. Find : $\int \frac{\sin x}{\sin^3 x + \cos^3 x} dx$. R&U [S.Q.P. Dec. 2016-17]

[HOTS]

Sol. Let $I = \int \frac{\sin x}{\sin^3 x + \cos^3 x} dx$

$= \int \frac{\tan x \sec^2 x}{\tan^3 x + 1} dx$ ½

On substituting $\tan x = t$ and $\sec^2 x \; dx = dt$, we get 1

$I = \int \frac{t}{t^3 + 1} dt$

$= \int \frac{t}{(t+1)(t^2 - t + 1)} dt$

$= -\frac{1}{3}\int \frac{1}{t+1} dt + \frac{1}{3}\int \frac{t+1}{t^2 - t + 1} dt$ ½

$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\int \frac{(2t-1)+3}{t^2 - t + 1} dt$ 1

$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\int \frac{2t-1}{t^2-t+1} dt + \frac{1}{2}\int \frac{1}{t^2-t+1} dt$

$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2 - t + 1|$

$+ \frac{1}{2}\int \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt$

$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2 - t + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)$ 1

$= -\frac{1}{3}\log|\tan x + 1| + \frac{1}{6}\log|\tan^2 x - \tan x + 1|$

$+ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + C$ 1

[CBSE Marking Scheme 2016] (Modified)

Q. 2. Find : $\int \frac{\sqrt{x^2+1}\{\log(x^2+1) - 2\log x\}}{x^4} dx$.

A I A [NCERT] [O.D. Set I, II, III Comptt. 2014]

Sol. $\int \frac{\sqrt{x^2+1}\{\log(x^2+1) - 2\log x\}}{x^4} dx$

$= \int \sqrt{1 + \frac{1}{x^2}}\left(\log\left(1 + \frac{1}{x^2}\right)\right)\frac{1}{x^3} dx$ 1

Let, $1 + \frac{1}{x^2} = t^2$

or $\frac{-2}{x^3} dx = 2\,t dt$

or $\frac{1}{x^3} dx = -\,t dt$ ½

$= -\int t(2\log t) t \; dt = -2\int \log t \cdot t^2 dt$ 1

$= -2\log t \cdot \frac{t^3}{3} + \int 2 \frac{1}{t} \cdot \frac{t^3}{3} dt$ 1

$= -\frac{2}{3}\log t \cdot t^3 + \frac{2}{9}t^3 + C$ ½

$= -\frac{1}{3}\left(1 + \frac{1}{x^2}\right)^{3/2}\left(\log\left(1 + \frac{1}{x^2}\right) - \frac{2}{3}\right) + C$ 1

[CBSE Marking Scheme 2014] (Modified)

Q. 3. Find : $\int \frac{1}{\cos^4 x + \sin^4 x} dx$.

A I A [O.D. Set I, 2014][HOTS]

Sol. $I = \int \frac{1}{\cos^4 x + \sin^4 x} dx$

Dividing numerator and denominator by $\cos^4 x$,

$= \int \frac{\sec^4 x}{1 + \tan^4 x} dx$

$= \int \frac{(1 + \tan^2 x)\sec^2 x}{1 + \tan^4 x} dx$ 1½

Putting, $\tan x = t$

$\Rightarrow \sec^2 x \; dx = dt$

$= \int \frac{(t^2 + 1)dt}{t^4 + 1}$

$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt$ {dividing by t^2} 1

$= \int \frac{dz}{z^2 + (\sqrt{2})^2}$, where $t - \frac{1}{t} = z$ 1

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + C$ ½

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) + C$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C$ 1

[CBSE Marking Scheme 2014] (Modified)

Q. 4. Evaluate : $\int\left(\sqrt{\cot x} + \sqrt{\tan x}\right) dx$. A I A [NCERT]

[O.D. Set I, 2014]

Sol. $I = \int\left(\sqrt{\cot x} + \sqrt{\tan x}\right) dx$

$$= \int \frac{\cos x + \sin x}{\sqrt{\sin x \cos x}} \cdot dx \qquad 1$$

Putting $\sin x - \cos x = t$, so that $(\cos x + \sin x)\, dx = dt$

and $\sin x . \cos x = \frac{1}{2}(1 - t^2)$ 1

$$I = \sqrt{2} \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2} \sin^{-1} t + C$$

$$= \sqrt{2} \sin^{-1}(\sin x - \cos x) + C \qquad 3$$

[CBSE Marking Scheme 2014] (Modified)

Detailed Solution :

$$= \int \left(\sqrt{\cot x} + \sqrt{\tan x}\right) dx$$

$$= \int \left(\sqrt{\cot x} + \frac{1}{\sqrt{\cot x}}\right) dx \qquad \left[\text{Using } \tan x = \frac{1}{\cot x}\right]$$

$$= \int \left(\frac{\cot x + 1}{\sqrt{\cot x}}\right) dx$$

$$I = \int \sqrt{\tan x}\,(\cot x + 1)\, dx \qquad \text{Put } \tan x = t^2$$

Let $\tan x = t^2$

$$\sec^2 x\, dx = 2t\, dt$$

$$1 + \tan^2 x = 2t \frac{dt}{dx}$$

$$1 + t^4 = 2t \frac{dt}{dx}$$

$$dx = \frac{2t}{(1+t^4)} dt$$

$$\text{Now, I} = \int \sqrt{t^2}\left(\frac{1}{t^2} + 1\right)\left(\frac{2t}{1+t^4}\right) dt$$

$$= \int t\left(\frac{1}{t^2} + 1\right)\left(\frac{2t}{1+t^4}\right) dt$$

$$= \int 2t^2\left(\frac{1+t^2}{t^2}\right)\left(\frac{1}{1+t^4}\right) dt$$

$$= 2\int \left(\frac{1+t^2}{1+t^4}\right) dt$$

$$= 2\int \frac{\frac{1+t^2}{t^2}}{\frac{1+t^4}{t^2}} dt$$

$$= 2\int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 2 - 2} dt$$

$$I = 2\int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + \left(\sqrt{2}\right)^2} dt$$

$$\text{Let } t - \frac{1}{t} = y \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dy$$

$$\therefore \quad I = 2\int \frac{dy}{y^2 + \left(\sqrt{2}\right)^2}$$

$$= 2\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right)\right] + C$$

$$= \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right) + C$$

$$= \sqrt{2} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}\, t}\right) + C$$

$$= \sqrt{2} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\sqrt{\tan x}}\right) + C$$

$$= \sqrt{2} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2 \tan x}}\right) + C$$

Concept Applied

- Second fundamental theorem
- Properties of definite integral

Commonly Made Error

- Sometimes candidates make errors in substitution and simplification which leads errors in further simplification.

Answering Tip

- Learn to substitute and simplify trigonometric equations.

Topic-2 Definite Integral

Concepts Covered • *Second fundamental theorem,* • *Properties of definite integral.*

Revision Notes

➢ **Meaning of Definite Integral of Function**

If $\int f(x)dx = F(x)$ *i.e.*, $F(x)$, be an integral of $f(x)$, then $F(b) - F(a)$ is called the definite integral of $f(x)$ between the limits a and b and in symbols it is written as

$\int_a^b f(x)dx = [F(x)]_a^b$. Moreover, the definite integral gives a unique and definite value (numeric value) of **anti-derivative** of the function between the given intervals. It acts as a substitute for evaluating the area analytically.

Key Word

Anti-derivative: In calculus, an anti-derivative, inverse derivative, primitive function, primitive integral or indefinite integral of a function f is a differentiable function F whose derivative is equal to the original function f. This can be stated symbolically as $F' = f$.

Key Formulae

(a) $\int_a^b f(x)dx = F(b) - F(a)$

(b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(c) $\int_a^b f(x)dx = \int_b^a f(t)dt \quad (dx = dt)$

(d) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

(e) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(f) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

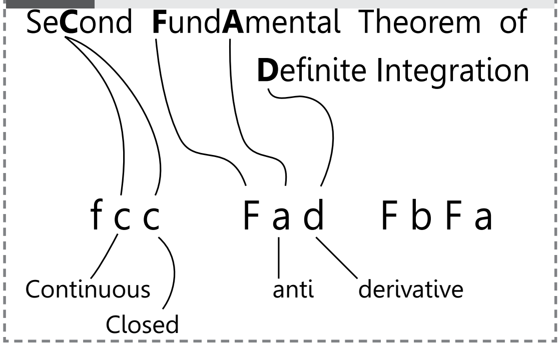
(g) $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is an even function } i.e., f(-x) = f(x) \\ 0, \text{ if } f(x) \text{ is an odd function } i.e., f(-x) = -f(x) \end{cases}$

(h) $\int_{-a}^{a} f(x)dx = \int_0^a \{f(x) + f(-x)\}dx$

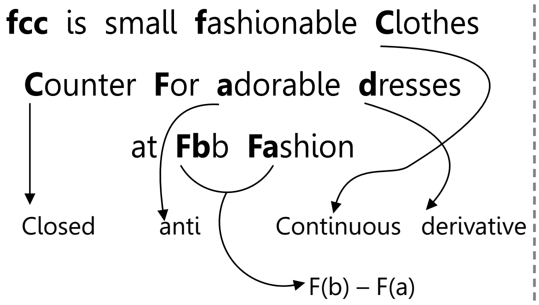
(i) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \\ 0 \quad ; \text{ if } f(2a-x) = -f(x) \end{cases}$

(j) $\int_0^{2a} f(x)dx = \int_0^{2a} \{f(x) + f(2a-x)\}dx$

Mnemonics



You can also remember



Interpretation :

Let f be a continuous function defined on a closed interval $[a,b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$, where a and b are called limit of Integration.

Example 1

Evaluate the following definite integral :

$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx.$

Solution:

Step I : Let

$$I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$$

$$= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$$

$$= I_1 + I_2$$

Step II :

$I_1 = 0$ (being an odd function)

Step III :

$$I_2 = 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$$

(being an even function)

$\therefore \quad I = I_2$

$$= 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$$

$$= 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

Step IV :

Let $I_3 = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

Apply the property $\int_0^a f(x) = \int_0^a f(a-x)dx$,

$$= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$$

$$= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

$$= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx - I_3$$

Step V :

$\therefore \quad 2I_3 = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$

Putting $\cos x = t$ or $-\sin x\, dx = dt$

When $x = 0$; $t = 1$ & $x = \pi$; $t = -1$

$$2I_3 = -\pi\int_1^{-1} \frac{dt}{1+t^2}$$

$$= \pi\left[\tan^{-1} x\right]_{-1}^{1} \qquad \left[\because \int_0^b f(x)dx = \int_0^a f(x)dx\right]$$

$$= \frac{\pi^2}{2}$$

or $\quad I_3 = \frac{\pi^2}{4}$

Hence $\quad I = \pi^2 \qquad [\because I_3 = 4I]$

OBJECTIVE TYPE QUESTIONS

A Multiple Choice Questions

Q. 1. The value of $\int_{-\pi/2}^{\pi/2}(x^3 + x\cos x + \tan^5 x + 1)\,dx$ is

(A) 0 **(B)** 2

(C) π **(D)** 1

Ans. Option (C) is correct.

Explanation: Let,

$$I = \int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)dx$$

$$= \int_{-\pi/2}^{\pi/2} x^3 dx + \int_{-\pi/2}^{\pi/2} x\cos x + \int_{-\pi/2}^{\pi/2} \tan^5 x dx + \int_{-\pi/2}^{\pi/2} 1\cdot dx$$

It is known that if $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

and if $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)dx = 0$$

$$\therefore\ I = 0+0+0+2\int_0^{\pi/2} 1\cdot dx$$

$$= 2[x]_0^{\pi/2} = \frac{2\pi}{2} = \pi$$

Q. 2. If $f(a+b-x) = f(x)$, then $\int_a^b xf(x)dx$ is equal to

(A) $\frac{a+b}{2}\int_a^b f(b-x)dx$ **(B)** $\frac{a+b}{2}\int_a^b f(b+x)dx$

(C) $\frac{b-a}{2}\int_a^b f(x)dx$ **(D)** $\frac{a+b}{2}\int_a^b f(x)dx$

Ans. Option (D) is correct.

Explanation:

Let $\quad I = \int_a^b xf(x)dx \qquad$ ...(i)

$$I = \int_a^b (a+b-x)f(a+b-x)dx$$

$$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$\Rightarrow \quad I = \int_a^b (a+b-x)f(x)dx$

$\Rightarrow \quad I = (a+b)\int_a^b f(x)dx - I \qquad$ [Using Eq. (i)]

$\Rightarrow \quad I + I = (a+b)\int_a^b f(x)dx$

$\Rightarrow \quad 2I = (a+b)\int_a^b f(x)dx$

$\Rightarrow \quad I = \left(\frac{a+b}{2}\right)\int_a^b f(x)dx$

Q. 3. If $f(x) = \int_0^x t\sin t\,dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$ **(B)** $x\sin x$

(C) $x\cos x$ **(D)** $\sin x + x\cos x$

Ans. Option (B) is correct.

Explanation:

$$f(x) = \int_0^x t \sin t dt$$

Integrating by parts, we obtain

$$f(x) = t\int_0^x \sin t dt - \int_0^x \left\{\left(\frac{d}{dt}t\right)\int \sin t dt\right\} dt$$

$$= [t(-\cos t)]_0^x - \int_0^x (-\cos t) dt$$

$$= [-t\cos t + \sin t]_0^x$$

$$= -x\cos x + \sin x$$

$$\Rightarrow \quad f'(x) = -[\{x(-\sin x)\} + \cos x] + \cos x$$

$$= x\sin x - \cos x + \cos x$$

$$= x\sin x$$

Q. 4. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ **is equal to**

(A) 1 **(B)** 2

(C) 3 **(D)** 4

Ans. Option (A) is correct.

Explanation: Let

$$I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$$

$$= \int_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x}$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x dx$$

$$= \int_0^{\pi/4} \sec^2 x dx$$

as $\sec^2 x$ is an even function

$$= [\tan x]_0^{\pi/4}$$

$$= 1$$

Q. 5. $\int_0^{\pi/2} \sqrt{1-\sin 2x}\, dx$ **is equal to**

(A) $2\sqrt{2}$ **(B)** $2(\sqrt{2}+1)$

(C) 2 **(D)** $2(\sqrt{2}-1)$

Ans. Option (D) is correct.

Explanation: Let

$$I = \int_0^{\pi/2} \sqrt{1-\sin 2x} dx$$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}\, dx + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2}\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2$$

$$= 2(\sqrt{2}-1)$$

Q. 6. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ **equals**

(A) $\frac{\pi}{3}$ **(B)** $\frac{2\pi}{3}$

(C) $\frac{\pi}{6}$ **(D)** $\frac{\pi}{12}$

Ans. Option (D) is correct.

Explanation: Let

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Q. 7. $\int_0^{2/3} \frac{dx}{4+9x^2}$ **equals**

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{12}$

(C) $\frac{\pi}{24}$ **(D)** $\frac{\pi}{4}$

Ans. Option (C) is correct.

Explanation:

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put $3x = t$

$\Rightarrow$ $3dx = dt$

$$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+(t)^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_0^{2/3} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left[\frac{3}{2}\times\frac{2}{3}\right] - \frac{1}{6}\tan^{-1}(0)$$

$$= \frac{1}{6}\tan^{-1}(1)$$

$$= \frac{1}{6}\tan^{-1}\left[\tan\frac{\pi}{4}\right]$$

$$= \frac{\pi}{24}$$

Q. 8. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$, **where $x = 0$, is equal to**

(A) – 2 (B) 0
(C) 1 (D) π

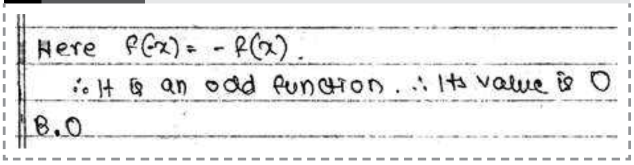
[CBSE Foreign Set II, 2020]

Ans. Option (B) is correct.
Explanation: 0

[CBSE Marking Scheme 2020]

OR

Topper Answer, 2020


Here f(-x) = - f(x).
∴ It is an odd function. ∴ Its value is 0
B.0

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Evaluate $\int_0^{2\pi} |\sin x|\,dx$.

R&U [CBSE OD Set-I,II,III 2020]

Sol. $I = 4\int_0^{\frac{\pi}{2}} \sin x\,dx = 4$ ½+½

[CBSE Marking Scheme 2020]

Detailed Solution:

Let
$$I = \int_0^{2\pi} |\sin x|\,dx$$
$$= \int_0^{\pi} |\sin x|\,dx + \int_{\pi}^{2\pi} |\sin x|\,dx$$
$$= \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx$$
$$= [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi}$$
$= [-\cos \pi + \cos 0] - [-\cos 2\pi + \cos \pi]$
$= [1 + 1] - [-1 - 1] = 2 + 2 = 4$

Commonly Made Error

- Mostly students fail in correctly splitting the modulus function involving trigonometric functions.

Answering Tip

- Practice problems in modulus functions and trigonometric functions.

AI **Q. 2. If** $\int_0^{a} \frac{dx}{1+4x^2} = \frac{\pi}{8}$, **then find the value of *a*.**

R&U [NCERT EXEMPLAR]
[CBSE OD Set-I, II 2020]

Q. 3. If [*x*] denotes the greatest integer function, then find $\int_0^{3/2} [x^2]dx$ **.** AI R&U [CBSE OD Set-III 2020]

Sol.
$$\int_0^{3/2} [x^2]dx = \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{3/2} 2\,dx \quad ½$$
$$= 0 + \left(\sqrt{2}-1\right) + 2\left(\frac{3}{2} - \sqrt{2}\right)$$
$$= 2 - \sqrt{2} \quad ½$$

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int_0^{3/2} [x^2]dx$$

For $0 \leq x < 1, 0 \leq x^2 < 1$, hence $[x^2] = 0$

For $1 \leq x < \sqrt{2}, 1 \leq x^2 < 2$, hence $[x^2] = 1$

For $\sqrt{2} \leq x < \frac{3}{2}, 2 \leq x^2 < \frac{9}{4}$, hence $[x^2] = 2$

$$\int_0^{3/2} [x^2]dx = \int_0^1 0dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{3/2} 2dx$$
$$= 0[x]_0^1 + 1[x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{3/2}$$
$$= 0 + \sqrt{2} - 1 + 3 - 2\sqrt{2}$$
$$= 2 - \sqrt{2}$$

Q. 4. Find the value of $\int_1^4 |x-5|\,dx$.

U [CBSE Delhi Set I, II, III-2020]

Sol. $\int_1^4 |x-5|\,dx = \int_1^4 (5-x)\,dx = \frac{15}{2}$ ½+½

[CBSE Marking Scheme, 2020]

Detailed Solution:

$$\int_1^4 |x-5|\,dx = \int_1^4 -(x-5)dx$$

$$= -\frac{1}{2}[(x-5)^2]_1^4$$

$$\int_1^4 |x-5|\,dx = -\frac{1}{2}[1-16] = \frac{15}{2}$$

Q. 5. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \sin x\, dx$

A I R&U [CBSE SQP 2020-21]

Sol. $\because f(x)$ is an odd function

$$\therefore \int_{-\pi/2}^{\pi/2} x^2 \sin x\, dx = 0$$ 1

[CBSE Marking Scheme, 2020]

Detailed Answer :

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \sin x dx = 0$$

$$f(x) = x^2 \sin x$$

replace x by $-x$ we get

$$f(-x) = (-x)^2 \sin(-x)$$

$f(-x) = -x^2 \sin x$ this implies $f(x)$ is an odd function

$$\left[\therefore \int_{-a}^{a} f(x)dx = 0, \text{ if } f(x) \text{ is an odd function } i.e., f(-x) = -f(x)\right]$$

Commonly Made Error

- Some students apply by parts and make it more complicated.

Answering Tip

- Apply the properties to easily arrive at the answer.

Q. 6. Evaluate : $\int_{-2}^{2}(x^3+1)dx.$ R [CBSE SQP-2020-21]

Sol. $\int_{-2}^{2}(x^3+1)dx = \int_{-2}^{2}(x^3)dx + \int_{-2}^{2}1\,dx = I_1 + I_2$ ½

$= 0 + [x]_{-2}^{2}$

(As I_1 is odd function)

$= 2 + 2$

$= 4$ ½

[CBSE SQP Marking Scheme 2020]

Q. 7. Evaluate : $\int_2^3 3^x\, dx.$ R&U [Delhi 2017]

Sol. $\int_2^3 3^x dx = \left[\frac{3^x}{\log 3}\right]_2^3 = \frac{18}{\log 3}.$ 1

[CBSE Marking Scheme, 2017]

Q. 8. Evaluate : $\int_0^{2\pi} \cos^5 x\, dx.$ R&U [Foreign 2017]

Concept Applied

Given load function

Q. 9. Evaluate : $\int_1^3 |2x-1|\, dx$ [Board, 2020]

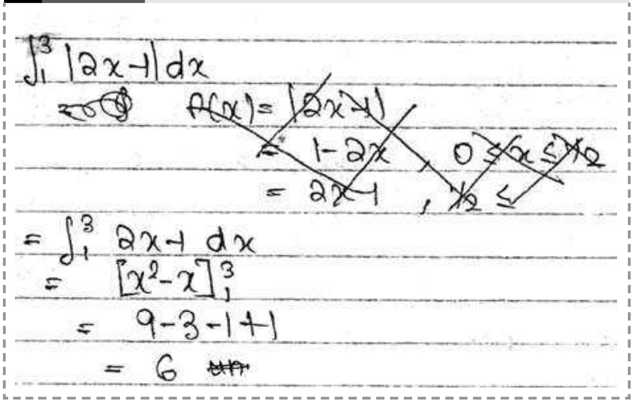
Sol. $\int_1^3 |2x-1|\,dx = \int_1^3 (2x-1)dx$

$$= \left[\frac{1}{4}(2x-1)^2\right]_1^3 = 6$$

[CBSE Marking Scheme, 2020]

OR

Topper Answer, 2020


$\int_1^3 |2x-1|\,dx$

$= \int_1^3 2x-1\,dx$

$= [x^2-x]_1^3$

$= 9-3-1+1$

$= 6$

Short Answer Type Questions-I (2 marks each)

Q. 1. Evaluate : $\int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2}\right] e^{2x} dx$

R&U [CBSE Delhi Set-I, II, III 2020]

Q. 2. Find the value of $\int_0^1 x(1-x)^n dx$.

A I R&U [CBSE Delhi Set-I, II, III 2020]

[CBSE SQP 2020-21]

Sol. $\int_0^1 x(1-x)^n dx = \int_0^1 (1-x)(1-1+x)^n dx$ ½

$= \int_0^1 (x^n - x^{n+1}) dx$ ½

$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$

$= \frac{1}{n+1} - \frac{1}{n+2}$ ½

or $\frac{1}{(n+1)(n+2)}$ ½

[CBSE Marking Scheme 2020]

Detailed Solution:

Suppose

$$I = \int_0^1 x(1-x)^n dx$$

$$\Rightarrow I = \int_0^1 (1-x)[1-(1-x)]^n dx$$

$$\Rightarrow I = \int_0^1 (1-x)x^n dx$$

$$\Rightarrow I = \int_0^1 (x^n - x^{n+1}) dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$\Rightarrow I = \left[\frac{1}{n+1} - \frac{1}{n+2}\right] - [0-0] = \frac{1}{(n+1)(n+2)}$$

Q. 3. Find the value of $\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx$.

A **[CBSE Delhi Set- III 2020]**

Sol. $\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx$

$= \int_0^1 \tan^{-1}\left(\frac{(1-x)-x}{1+(1-x)x}\right)dx$

$= \int_0^1 \tan^{-1}(1-x)dx - \int_0^1 \tan^{-1}x\,dx$ 1

$= 0$

as $\int_0^1 \tan^{-1}x\,dx = \int_0^1 \tan^{-1}(1-x)dx$ 1

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{(1-x)-x}{1+x(1-x)}\right)dx$$

$$= \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}x\right]dx$$

$$= \int_0^1 \tan^{-1}(1-(1-x))dx - \int_0^1 \tan^{-1}x\,dx$$

[Because $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$$= \int_0^1 \tan^{-1}x\,dx - \int_0^1 \tan^{-1}x\,dx = 0$$

$$\therefore \int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx = 0$$

Commonly Made Error

- Some students directly apply the property without simplifying the inverse function.

Answering Tip

- Simplify inverse trigonometric and logarithmic functions before applying property.

Q. 4. Evaluate $\int_{-\pi}^{\pi}(1-x^2)\sin x.\cos^2 x.dx$

A I R **[CBSE Delhi Set-III 2019]**

Sol. Let $f(x) = (1-x^2).\sin x\cos^2 x$ 1

as $f(-x) = -f(x) \Rightarrow f$ is odd function. 1

$\therefore I = 0$

[CBSE Marking Scheme 2019]

Detailed Solution :

Let $f(x) = (1-x^2)\sin x\cos^2 x$

Then $f(-x) = [1-(-x)^2]\sin(-x)[\cos(-x)]^2$

$= (1-x^2)(-\sin x)\cos^2 x$

$= -(1-x^2)\sin x\cos^2 x$

$= -f(x)$

So, $f(x)$ is an odd function,

$$\int_{-\pi}^{\pi} f(x)dx = 0$$

$$\Rightarrow \int_{-\pi}^{\pi}(1-x^2)\sin x\cos^2 x\,dx = 0$$

Q. 5. Evaluate $\int_{-1}^{2}\frac{|x|}{x}dx$. R **[CBSE Delhi Set-III, 2019]**

Short Answer Type Questions-II (3 marks each)

Q. 1. Evaluate $\int_0^1 \sqrt{3-2x-x^2}\,dx$

R&U **[CBSE OD Set III-2020]**

Q. 2. Prove that : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, **hence evaluate** $\int_0^a \frac{x\sin x}{1+\cos^2 x}dx$.

A I U [CBSE Delhi Set-II, 2019]

OR

Evaluate : $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$.

A [Delhi 2017][NCERT] [OD Comptt. 2017] [CBSE OD Set I-2020]

Q. 3. Evaluate : $\int_1^3 |x^2-2x|dx$.

A I U [CBSE SQP-2020-21]

Sol. Consider $I = \int_1^3 |x^2-2x|\,dx$

$$|x^2-2x| = \begin{cases} -(x^2-2x) & \text{where } 1 \le x < 2 \\ (x^2-2x) & \text{where } 2 \le x \le 3 \end{cases}$$

$$I = \int_1^2 |x^2-2x|\,dx + \int_2^3 |x^2-2x|\,dx$$ **1**

$$I = \int_1^2 -(x^2-2x)\,dx + \int_2^3 (x^2-2x)\,dx$$ **1**

$$I = -\left[\frac{x^3}{3}-x^2\right]_1^2 + \left[\frac{x^3}{3}-x^2\right]_2^3$$

$$I = -\left[\frac{8}{3}-4-\frac{1}{3}+1\right]+\left[9-9-\frac{8}{3}+4\right]$$

$$= -\frac{7}{3}+3-\frac{8}{3}+4$$

$$I = \frac{6}{3} = 2$$ **1**

[CBSE SQP Marking Scheme 2020] (Modified)

Commonly Made Error

- Most of the students carelessly neglect the modulus sign or are unable to change the modulus.

Answering Tip

- Learn comprehensively all standard methods of integrals, problems based on them, definite integrals, their basic properties and the problems based on them.

Q. 4. Prove that : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **and hence evaluate** $\int_0^{\pi/2} \frac{x}{\sin x+\cos x}dx$

U [CBSE OD Set-I, 2019]

Sol.

$$\int_0^a f(x)dx = -\int_a^0 f(a-t)dx$$

Put $x = a-t, dx = -dt$

Upper limit $= t = a - x = a - a = 0$

Lower limit $= t = a - x = a - 0 = a$

$$= \int_0^a f(a-t)dt = \int_0^a f(a-x)dx$$ ½

Let $I = \int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x}dx$ ...(i)

$$I = \int_0^{\pi/2} \frac{\pi/2-x}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\therefore I = \int_0^{\pi/2} \frac{\frac{\pi}{2}-x}{\cos x+\sin x}dx$$ ...(ii) **1**

Adding, (i) and (ii), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2}\frac{1}{\cos x+\sin x}dx = \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\frac{1}{\cos\left(\frac{x-\pi}{4}\right)}dx$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\sec\left(x-\frac{\pi}{4}\right)dx$$ ½

$$\Rightarrow 2I = \frac{\pi}{2\sqrt{2}}\left\{\log\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|\right\}_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2\sqrt{2}}\{\log(\sqrt{2}+1)-\log(\sqrt{2}-1)\}$$ ½

$$\Rightarrow I = \frac{\pi}{4\sqrt{2}}\{\log(\sqrt{2}+1)-\log(\sqrt{2}-1)\} \text{ or}$$

$$\frac{\pi}{4\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$ ½

[CBSE Marking Scheme, 2019] (Modified)

Q. 5. Evaluate : $\int_{-1}^1 \frac{x+|x|+1}{x^2+2|x|+1}.dx$.

R&U [S.Q.P. 2018-19]

Sol. $$I = \int_{-1}^1 \frac{x+|x|+1}{x^2+2|x|+1}$$

$$= \int_{-1}^1 \frac{x}{x^2+2|x|+1}dx + \int_{-1}^1 \frac{|x|+1}{x^2+2|x|+1}dx$$ ½

$= I_1 + I_2$ (Say) ...(1)

Now, $I_1 = \int_{-1}^{1} \frac{x}{x^2+2|x|+1}dx$

Let $f(x) = \frac{x}{x^2+2|x|+1}dx$

$f(-x) = \frac{-x}{(-x)^2+2|-x|+1} = \frac{-x}{x^2+2|x|+1} = -f(x)$

$\because f(x)$ is odd function. **1**

Hence, $I_1 = 0$ ...(2)

Also, $I_2 = \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}dx$

Let $g(x) = \frac{|x|+1}{x^2+2|x|+1}$

$\Rightarrow$ $g(-x) = \frac{|-x|+1}{(-x)^2+2|-x|+1}$

$g(-x) = \frac{|x|+1}{x^2+2|x|+1} = g(x)$

$\because$ $g(x)$ is even function **1**

$\therefore$ $I_2 = 2\int_0^1 \frac{x+1}{x^2+2x+1} = 2\int_0^1 \frac{1}{x+1}dx$

$= 2\left[\log|x+1|\right]_0^1$

$= 2[\log 2 - \log 1]$

$I_2 = 2\log 2$ ...(3)

From (1), (2) and (3), we get

$I = 2\log 2$ ½

[CBSE Marking Scheme 2018] (Modified)

Q. 6. Evaluate : $\int_0^{\frac{3}{2}} |x\sin\pi x|\,dx$. R&U [Delhi 2017]

Sol. $I = \int_0^{\frac{3}{2}} |x\sin\pi x|\,dx$

$= \int_0^1 x\sin\pi x.dx - \int_1^{\frac{3}{2}} x\sin\pi x\, dx$ **1**

$= \left[-x\frac{\cos x\pi}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_0^1 - \left[-\frac{x\cos x\pi}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_1^{\frac{3}{2}}$ **1**

$= \frac{2}{\pi} + \frac{1}{\pi^2}$ **1**

[CBSE Marking Scheme, 2017] (Modified)

Q. 7. $\int_1^4 \{|x-1|+|x-2|+|x-4|\}dx$ R&U [O.D., 2017]

Sol. $I = \int_1^4 \{|x-1|+|x-2|+|x-4|\}dx$

$= \int_1^4 (x-1)dx - \int_1^2 (x-2)dx + \int_2^4 (x-2)dx - \int_1^4 (x-4)dx$ **1**

$= \left.\frac{(x-1)^2}{2}\right]_1^4 - \left.\frac{(x-2)^2}{2}\right]_1^2 + \left.\frac{(x-2)^2}{2}\right]_2^4 - \left.\frac{(x-4)^2}{2}\right]_1^4$ **1**

$= \frac{9}{2} + \frac{1}{2} + 2 + \frac{9}{2} = 11\frac{1}{2}$ or $\frac{23}{2}$ **1**

[CBSE Marking Scheme, 2017] (Modified)

OR

Topper Answer, 2017

$f(x) = |x-1| + |x-2| + |x-4|$

$= \begin{cases} (x-1) - (x-2) - (x-4) & ; 1 \le x < 2 \\ (x-1) + (x-2) - (x-4) & ; 2 \le x \le 4 \end{cases}$

1.5
+ve −ve −ve
3
+ve +ve −ve

$= \begin{cases} x-1-x+2-x+4 & ; 1 \le x < 2 \\ x-1+x-2-x+4 & ; 2 \le x \le 4 \end{cases}$

$= \begin{cases} 5-x & ; 1 \le x < 2 \\ x+1 & ; 2 \le x \le 4 \end{cases}$

$$= \begin{cases} (x-1)-(x-2)-(x-4) \ ; & 1 \le x < 2 \quad +ve \ -ve \ -ve \\ (x-1)+(x-2)-(x-4) \ ; & 2 \le x \le 4 \quad +ve \ +ve \ -ve \end{cases}$$

3

$$= \left[10-2\right]-\left[5-\frac{1}{2}\right]+\left[8+4\right]-\left[2+2\right]$$

$$= 8-\frac{9}{2}+12-4$$

$$= 16-\frac{9}{2}$$

$$= \frac{23}{2} = 11.5$$

AI Q. 8. Evaluate : $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}dx$.

A [OD 2017] [NCERT]
[Delhi Set I, II, III Comptt. 2016]
[OD Comptt. 2017]

Sol. Let, $I = \int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}$ ...(i)

or $I = \int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}$ ½

$= \int_0^{\pi} \frac{(\pi-x)\cdot\tan x}{\sec x+\tan x}dx$ ...(ii) ½

Add eqn. (i) & (ii),

$2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x+\tan x}dx$

Multiply & divide by (sec x – tan x) on RHS, ½

or $2I = \pi\int_0^{\pi} \tan x(\sec x-\tan x)dx$

$= \pi\int_0^{\pi} (\sec x\cdot\tan x-\sec^2 x+1)dx$ ½

or $2I = \left[\pi(\sec x-\tan x+x)\right]_0^{\pi}$

$= \pi\{(-1-0+\pi-1)\} = \pi(\pi-2)$ ½

$\therefore$ $I = \frac{\pi^2-2\pi}{2}$ ½

[CBSE Marking Scheme 2017] (Modified)

AI Q. 9. Evaluate : $\int_{-1}^{2}|x^3-x|\,dx$. R&U [NCERT]

[CBSE OD SET II 2020] [Delhi Set I, II, III 2016]

Sol. $\because$ $\int_{-1}^{2}|x^3-x|\,dx = \int_{-1}^{0}(x^3-x)dx$

$-\int_0^1(x^3-x)dx+\int_1^2(x^3-x)dx$ **1**

$= \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}-\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_0^1+\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2$ **1**

$= \left[(0-0)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]-\left[\left(\frac{1}{4}-\frac{1}{2}\right)-(0-0)\right]$

$+\left[(4-2)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$

$= \frac{1}{4}+\frac{1}{4}+2+\frac{1}{4} = \frac{11}{4}$ **1**

[CBSE Marking Scheme 2020] (Modified)

Long Answer Type Questions-I (4 marks each)

Q. 1. Evaluate : $\int_{-1}^{2}|x^3-3x^2+2x|\,dx$

[SQP 2021-2022]

Sol. The given definite integral

$= \int_{-1}^{2}|x(x-1)(x-2)|\,dx$

$= \int_{-1}^{0}|x(x-1)(x-2)|\,dx+\int_0^1|x(x-1)(x-2)|\,dx$

$+\int_1^2|x(x-1)(x-2)|\,dx$ **1½**

$= -\int_{-1}^{0}(x^3-3x^2+2x)dx+\int_0^1(x^3-3x^2+2x)\,dx$

$-\int_1^2(x^3-3x^2+2x)\,dx$ ½

$= -\left[\frac{x^4}{4}-x^3+x^2\right]_{-1}^{0}+\left[\frac{x^4}{4}-x^3+x^2\right]_0^1$

$-\left[\frac{x^4}{4}-x^3+x^2\right]_1^2$

$= \frac{9}{4}+\frac{1}{4}+\frac{1}{4} = \frac{11}{4}$ **2**

[CBSE Marking Scheme 2022]

Q. 2. Evaluate : $\int_0^{\frac{3}{2}} |x \cos \pi x| dx$

R&U [O.D. Set I, II, III 2016]

Sol. Let $I = \int_0^{\frac{3}{2}} |x \cos \pi x| dx$

In $0 \le x \le \frac{1}{2}$, $x \cos \pi x \ge 0$,

In $\frac{1}{2} \le x \le \frac{3}{2}$, $x \cos \pi x \le 0$,

$$\therefore \quad I = \int_0^{\frac{1}{2}} x \cos \pi x \, dx - \int_{\frac{1}{2}}^{\frac{3}{2}} x \cos \pi x \, dx$$

$$= \left[x \frac{(\sin \pi x)}{\pi} - \int (1). \frac{(\sin \pi x)}{\pi} dx \right]_0^{\frac{1}{2}} \quad \mathbf{1}$$

$$- \left[x \frac{(\sin \pi x)}{\pi} - \int (1). \frac{(\sin \pi x)}{\pi} dx \right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$\therefore \quad I = \left[\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{(\pi)^2} \right]_0^{\frac{1}{2}} - \left[\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \left[\frac{\frac{1}{2} \sin \frac{\pi}{2}}{\pi} + \frac{\cos \frac{\pi}{2}}{\pi^2} - 0 - \frac{\cos 0}{\pi^2} \right]$$

$$- \left[\frac{\frac{3}{2} \sin \frac{3\pi}{2}}{\pi} + \frac{\cos \frac{3\pi}{2}}{\pi^2} - \frac{\frac{1}{2} \sin \frac{\pi}{2}}{\pi} - \frac{\cos \frac{\pi}{2}}{\pi^2} \right] \quad \mathbf{1}$$

$$\therefore \quad I = \left[\frac{1}{2\pi} - \frac{1}{\pi^2} \right] - \left[\frac{3}{2\pi}(-1) + 0 - \frac{1}{2\pi} - 0 \right]$$

$$= \left[\frac{1}{2\pi} - \frac{1}{\pi^2} + \frac{3}{2\pi} + \frac{1}{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{5}{2} - \frac{1}{\pi} \right] \quad \mathbf{1}$$

$$\therefore \quad I = \frac{1}{\pi} \left[\frac{5}{2} - \frac{1}{\pi} \right]$$

[CBSE Marking Scheme 2016] (Modified)

Q. 3. Evaluate : $\int_0^{\pi} e^{2x} \sin \left(\frac{\pi}{4} + x \right) dx.$

A [Delhi Set I, II, III 2016]

Sol. $\int_0^{\pi} e^{2x} \sin \left(\frac{\pi}{4} + x \right) dx$

Let $I = \int e^{2x} . \sin \left(\frac{\pi}{4} + x \right) dx$

Integrating by parts, without limits

$$I = \sin \left(\frac{\pi}{4} + x \right) . \frac{e^{2x}}{2} - \int \cos \left(\frac{\pi}{4} + x \right) . \frac{e^{2x}}{2} dx$$

$$= \sin \left(\frac{\pi}{4} + x \right) . \frac{e^{2x}}{2} - \frac{1}{2} \int e^{2x} . \cos \left(\frac{\pi}{4} + x \right) dx$$

$$= \sin \left(\frac{\pi}{4} + x \right) . \frac{e^{2x}}{2} - \frac{1}{2} \left[\cos \left(\frac{\pi}{4} + x \right) . \frac{e^{2x}}{2} - \int - \sin \left(\frac{\pi}{4} + x \right) . \frac{e^{2x}}{2} dx \right] \quad \mathbf{1}$$

$$= \sin \left(\frac{\pi}{4} + x \right) . \frac{e^{2x}}{2} - \frac{1}{4} e^{2x} \cos \left(\frac{\pi}{4} + x \right) - \frac{1}{4} \int e^{2x} . \sin \left(\frac{\pi}{4} + x \right) dx$$

or $I \left(1 + \frac{1}{4} \right) = \frac{1}{4} e^{2x} \left\{ 2 \sin \left(\frac{\pi}{4} + x \right) - \cos \left(\frac{\pi}{4} + x \right) \right\}$

or $I = \frac{1}{5} e^{2x} \left\{ 2 \sin \left(\frac{\pi}{4} + x \right) - \cos \left(\frac{\pi}{4} + x \right) \right\}$ **½**

Now, $\int_0^{\pi} e^{2x} . \sin \left(\frac{\pi}{4} + x \right) dx$

$$= \frac{1}{5} \left[e^{2x} \left\{ 2 \sin \left(\frac{\pi}{4} + x \right) - \cos \left(\frac{\pi}{4} + x \right) \right\} \right]_0^{\pi}$$

$$= \frac{1}{5} \left[e^{2\pi} \left\{ 2 \sin \left(\pi + \frac{\pi}{4} \right) - \cos \left(\pi + \frac{\pi}{4} \right) \right\} - 2 \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right] \quad \mathbf{½}$$

$$= \frac{1}{5} \left[e^{2\pi} \left(-2 \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right]$$

$$= \frac{1}{5} \left[- \frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right]$$

$$= \frac{-1}{5\sqrt{2}} \left[e^{2\pi} + 1 \right] \quad \mathbf{1}$$

[CBSE Marking Scheme 2016] (Modified)

AI **Q. 4. Show that :** $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log \left(\sqrt{2} + 1 \right).$

[O.D. Set I, II, III 2016]

A [NCERT Exemplar]

Sol. $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ ...(i)

By applying property $\int_0^a f(x) = \int_0^a f(a-x)$

$$= \int_0^{\pi/2} \frac{\sin^2 \left(\frac{\pi}{2} - x \right)}{\sin \left(\frac{\pi}{2} - x \right) + \cos \left(\frac{\pi}{2} - x \right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx \quad \text{...(ii)}$$

Adding eqn. (i) & (ii)

$$\therefore \quad 2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\cos\left(x - \frac{\pi}{4}\right)} dx \qquad 1$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}} \left[\log \left| \sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right| \right]_0^{\pi/2} \qquad 1$$

$$= \frac{1}{\sqrt{2}} \left[\log|\sqrt{2}+1| - \log|\sqrt{2}-1| \right]$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right| \qquad ½$$

$$I = \frac{2}{2\sqrt{2}} \log|\sqrt{2}+1|$$

$$I = \frac{1}{\sqrt{2}} \log|\sqrt{2}+1| \qquad ½$$

[CBSE Marking Scheme 2016] (Modified)

Q. 5. Evaluate : $\int_0^{\frac{\pi}{4}} \log[1+\tan x]dx.$ A I A [NCERT]

[O.D. Set I, II, III Comptt. 2015]

Q. 6. Find : $\int \frac{2x+1}{\sqrt{3+2x-x^2}} dx$. [CBSE Foreign Set II]

Sol. $\int \frac{2x+1}{\sqrt{3+2x-x^2}} dx$

$$= -\int \frac{2-2x}{\sqrt{3+2x-x^2}} dx + 3\int \frac{1}{\sqrt{2^2-(x-1)^2}} dx$$

$$= -2\sqrt{3+2x-x^2} + 3\sin^{-1}\left(\frac{x-1}{2}\right) + C$$

[CBSE Marking Scheme 2022]

OR

Topper Answer, 2020

$\int \frac{2x+1\, dx}{\sqrt{3+2x-x^2}}$,

Now $2x+1 = A(2-2x)+B$

$2x+1 = x(-2A) + B+2A$.

$A = -1 \quad B = 3$

$I = -1\int \frac{2-2x}{\sqrt{3+2x-x^2}} dx + 3\int \frac{dx}{\sqrt{3+2x-x^2}}$

Now let $3+2x-x^2 = z^2$ and

$(2-2x)dx = 2z\,dz$

$3+2x-x^2$

$= -(x^2-2x-3)$

$= -(x^2-2.x+1-4)$

$= 2^2-(x-1)^2$

$\therefore I = -\int \frac{2z\,dz}{z} + 3\int \frac{dx}{\sqrt{2^2-(x-1)^2}}$

$= -2z + 3\left[\sin^{-1}\left(\frac{x-1}{2}\right)\right] + C$

$= -2\sqrt{3+2x-x^2} + 3\sin^{-1}\left(\frac{x-1}{2}\right) + C$, 'c' is integration constant

Long Answer Type Questions-II (5 marks each)

Q. 1. Evaluate : $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$

R&U [CBSE OD-18]

Q. 2. Evaluate the following : $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx$

R&U [SQP 2017-18]

Sol. The given definite integral

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{2-\cos 2x}dx + \frac{\pi}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2-\cos 2x}dx$$

$$f(x) = \frac{x}{2-\cos 2x}, f(-x) = \frac{-x}{2-\cos 2x} = -f(x)$$

Hence, f is odd. Therefore, $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{2-\cos 2x}dx = 0$ **1**

$$g(x) = \frac{1}{2-\cos 2x},\ g(-x)$$

$$= \frac{1}{2-\cos 2x} = g(x).$$

Hence, g is even. Thus

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2-\cos 2x}dx = 2\int_{0}^{\frac{\pi}{4}} \frac{1}{2-\cos 2x}dx$$ **1**

Hence, $I = \frac{\pi}{2}\int_{0}^{\frac{\pi}{4}} \frac{1}{2-\cos 2x}dx = \frac{\pi}{2}\int_{0}^{\frac{\pi}{4}} \frac{1}{1+2\sin^2 x}dx$

$$= \frac{\pi}{2}\int_{0}^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan^2 x + 2\tan^2 x}dx = \frac{\pi}{2}\int_{0}^{\frac{\pi}{4}} \frac{\sec^2 x}{1+3\tan^2 x}dx$$

$$= \frac{\pi}{2}\int_{0}^{1} \frac{1}{1+3t^2}dx \qquad [\tan x = t \text{ or } \sec^2 x\,dx = dt]$$ **2**

$$= \frac{\pi}{2}\times\frac{1}{3}\int_{0}^{1} \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2 + t^2}dt = \frac{\pi}{6}\sqrt{3}[\tan^{-1}\sqrt{3}t]_0^1$$

$$= \frac{\pi}{6}\sqrt{3}\left[\frac{\pi}{3}\right] = \frac{\sqrt{3}\pi^2}{18}$$ **1**

[CBSE Marking Scheme 2017-18] (Modified)

AI Q. 3. Evaluate the following : $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}dx$

A [S.Q.P. 2016-17]

Sol. $$I = \int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx,$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\cos^4 x + \sin^4 x}dx$$ **1**

$$2I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}\right)\cos x\sin x}{\cos^4 x + \sin^4 x}dx$$ **½**

$$= \left(\frac{\pi}{2}\right)\left[\int_0^{\frac{\pi}{4}} \frac{\cos x\sin x}{\cos^4 x + \sin^4 x}dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x\sin x}{\cos^4 x + \sin^4 x}dx\right]$$

$$= \left(\frac{\pi}{2}\right)\left[\int_0^{\frac{\pi}{4}} \frac{\tan x\sec^2 x}{1+\tan^4 x}dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\text{cosec}^2 x\cot x}{\cot^4 x + 1}dx\right]$$ **1**

$$\Rightarrow 2I = \left(\frac{\pi}{2}\right)\left[\int_0^1 \frac{1}{1+t^2}dx - \int_1^0 \frac{1}{1+p^2}dp\right]$$

substituting $\tan^2 x = t$, $2\tan x\sec^2 x dx = dt$

$\cot^2 x = p$, $-2\cot x\,\text{cosec}^2 x dx = dp$ **1**

or $$2I = \left(\frac{\pi}{4}\right)\left[\tan^{-1}t\right]_0^1 + \left(\frac{\pi}{4}\right)\left[\tan^{-1}p\right]_0^1$$

$$= \frac{\pi^2}{8}$$ **1**

or $$I = \frac{\pi^2}{16}$$ **½**

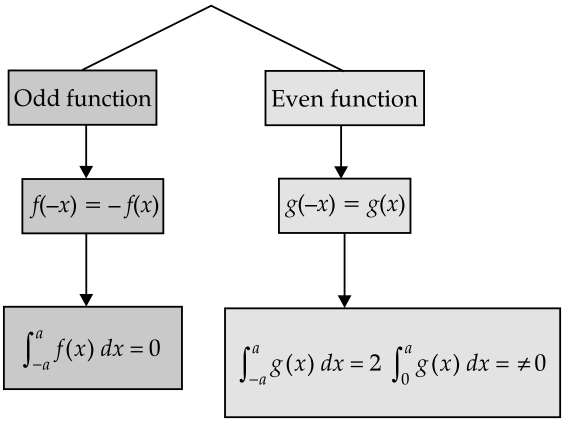
[CBSE Marking Scheme 2016] (Modified)

COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the below case and answer the questions that follow:



Q. 1. $\int_{-1}^{1} x^{99} dx =$ ______.

(A) 0 **(B)** 1
(C) –1 **(D)** 2

Ans. Option (A) is correct.

Explanation:

$$\int_{-1}^{1} x^{99} dx = 0, \text{ since } x^{99} \text{ is an odd function.}$$

Q. 2. $\int_{-\pi}^{\pi} x \cos x \, dx =$ ______.

(A) 1 **(B)** 0
(C) –1 **(D)** $\frac{\pi}{2}$

Ans. Option (B) is correct.

Explanation:

$$\int_{-\pi}^{\pi} x \cos x \, dx = 0, \text{ since } x \cos x \text{ is an odd function.}$$

Q. 3. $\int_{-\pi/2}^{\pi/2} \sin^3 x dx =$ ______.

(A) 1 **(B)** 0
(C) –1 **(D)** π

Ans. Option (B) is correct.

Explanation:

$$\int_{-\pi/2}^{\pi/2} \sin^3 x \, dx = 0, \text{ since } \sin^3 x \text{ is an odd function.}$$

Q. 4. $\int_{-\pi}^{\pi} x \sin x \, dx =$ ______.

(A) π **(B)** 0
(C) 2π **(D)** $\frac{\pi}{2}$

Ans. Option (C) is correct.

Explanation:

Since, $x \sin x$ is an even function

$$\begin{aligned} \int_{-\pi}^{\pi} x \sin x \, dx &= 2\int_{0}^{\pi} x \sin x \, dx \\ &= 2\left[-x\cos x + \int (1 \times \cos x \, dx)\right]_0^{\pi} \\ &= 2\left[-x\cos x + \sin x\right]_0^{\pi} \\ &= 2\left[(\pi + 0) - (0 + 0)\right] \\ &= 2\pi \end{aligned}$$

Q. 5. $\int_{-\pi}^{\pi} \tan x \sec^2 x dx =$ ______.

(A) 1 **(B)** –1
(C) 0 **(D)** 2

Ans. Option (C) is correct.

Explanation:

$$\int_{-\pi}^{\pi} \tan x \sec^2 x \, dx = 0, \text{ Since it is an odd function}$$

II. Read the below case and answer the questions that follow:

Today in class Mr. Sharma explained a particular form of integration, which help the students to get the answer of the given integral in that particular form. He said identify the particular form and apply the formula.

$$\begin{aligned} \int e^x[f(x) + f'(x)]dx &= \int e^x f(x)dx + \int e^x f'(x)dx \\ &= e^x f(x) + c \end{aligned}$$

Q. 1. $\int e^x(\sin x + \cos x)dx =$ ______.

(A) $e^x \cos x + c$ **(B)** $e^x \sin x + c$
(C) $e^x + c$ **(D)** $e^x (-\cos x + \sin x) + c$

Ans. Option (B) is correct.

Explanation:

$$\int e^x(\underbrace{\sin x}_{f(x)} + \underbrace{\cos x}_{f'(x)})dx = e^x \sin x + c$$

Q. 2. $\int e^x\left(\frac{x-1}{x^2}\right)dx =$ ______.

(A) $e^x + c$ **(B)** $\frac{e^x}{x} + c$

(C) $\frac{e^x}{x^2} + c$ **(D)** $\frac{-e^x}{x^2} + c$

Ans. Option (B) is correct.

Explanation:

$$\int e^x\left(\frac{x-1}{x^2}\right)dx = \int e^x\left(\underbrace{\frac{1}{x}}_{f(x)} - \underbrace{\frac{1}{x^2}}_{f'(x)}\right)dx = \frac{e^x}{x} + c$$

Q. 3. $\int e^x(x+1)dx =$ ______.

(A) $xe^x + c$ **(B)** $e^x + c$

(C) $e^{-x} + c$ **(D)** None of these

Ans. Option (A) is correct.

Explanation:

$$\int e^x(\underbrace{x}_{f(x)} + \underbrace{1}_{f'(x)})dx = xe^x + c$$

Q. 4. $\int_0^{\pi} e^x(\tan x + \sec^2 x)dx =$ ______.

(A) 0 **(B)** 1

(C) –1 **(D)** $-e^{\pi}$

Ans. Option (A) is correct.

Explanation:

$$\int_0^{\pi} e^x(\tan x + \sec^2 x)dx = \left[e^x \tan x\right]_0^{\pi} = 0$$

Q. 5. $\int \frac{xe^x}{(1+x)^2} dx =$ ______.

(A) $x e^x + c$ **(B)** $\frac{e^x}{(x+1)^2} + c$

(C) $\frac{x e^x}{x+1} + c$ **(D)** $\frac{e^x}{x+1} + c$

Ans. Option (D) is correct.

Explanation:

$$\int e^x\left[\frac{(x+1)-1}{(x+1)^2}\right]dx = \int e^x\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx$$

$$= \frac{e^x}{x+1}+c$$

III. Read the following text and answer the following questions on the basis of the same:

Reena and Sapna practice the problems based on integrals. They will try to evaluate the integrals based upon $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$.

Reena first explains the steps to solve this type of integrals.

Step 1: Obtain the integral, let it be $I = \int \frac{f'(x)}{f(x)}dx$

Step 2: Put $f(x) = t$ and replace $f'(x)dx$ by dt to obtain $I = \int \frac{1}{t}dt$

Step 3: Evaluate integral obtained in step II to obtain $I = \log|t| + c$

Step 4: Replace t by $f(x)$ step III to get $I = \log|f(x)| + c$

Q. 1. Evaluate: $\int \frac{2x+5}{x^2+5x-7}dx$

(A) $\log|(x^2 + 5x - 7)| + c$
(B) $\log|(2x + 5)| + c$
(C) $\log|(2x + 5)(x^2 + 5x - 7)| + C$
(D) 0

Ans. Option (A) is correct.

Explanation: Let $I = \int \frac{2x+5}{x^2+5x-7}dx$

let $x^2 + 5x - 7 = t$

$\Rightarrow d(x^2 + 5x - 7) = dt$

$\Rightarrow 2x + 5 = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c$

$= \log|(x^2 + 5x - 7)| + c$

Q. 2. Evaluate: $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

(A) $\log|e^x + e^{-x}| + c$ **(B)** $\log|e^x - e^{-x}| + c$
(C) $\log|e^{-x} - e^x| + c$ **(D)** $\log|e^x| + c$

Ans. Option (A) is correct.

Explanation: Let $I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

let $e^x + e^{-x} = t$

$\Rightarrow d(e^x + e^{-x}) = dt$

$\Rightarrow (e^x - e^{-x})dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c$

$= \log|(e^x + e^{-x})| + c$

Q. 3. Evaluate: $\int \frac{1}{x(3+\log x)}dx$

(A) $\log 3$ **(B)** $\log|(3 + \log x)| + c$
(C) $\log|x| + c$ **(D)** $\log|(x + 3)| + c$

Ans. Option (B) is correct.

Explanation: Let $I = \int \frac{1}{x(3+\log x)}dx$

let $3 + \log x = t$

$\Rightarrow d(3 + \log x) = dt$

$\Rightarrow \frac{1}{x}dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c$

$= \log|(3 + \log x)| + c$

Q. 4. Evaluate: $\int \frac{e^{2x}}{e^{2x}-2}dx$

(A) $\log|(e^{2x} - 2)| + c$ **(B)** $\log|e^{2x}| + c$
(C) $\frac{1}{2}\log|(e^{2x}-2)|+c$ **(D)** $\log 2$

Ans. Option (C) is correct.

Explanation: Let $I = \int \frac{e^{2x}}{e^{2x}-2}dx$

Now, let $e^{2x} - 2 = t$

$\Rightarrow d(e^{2x}) = dt$

$\Rightarrow 2e^{2x}dx = dt$

$\Rightarrow e^{2x}dx = \frac{dt}{2}$

$\Rightarrow I = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\log|t|+c$

$= \frac{1}{2}\log(e^{2x}-2)+c$

Q. 5. Evaluate: $\int \frac{1}{1+e^{-x}}dx$

(A) $e^{-x} + c$ **(B)** $\frac{1}{1+e^{-x}}+c$
(C) $\log|1 + e^{-x}| + c$ **(D)** $\log|1 + e^x| + c$

Ans. Option (D) is correct.

Explanation: Let $I = \int \frac{1}{1+e^{-x}}dx$

or $I = \int \frac{e^x}{e^x+1}dx$

(on dividing Nr and Dr by e^{-x})

let $e^x + 1 = t$

$\Rightarrow d(e^x + 1) = dt$

$\Rightarrow e^x dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c$

$= \log|1 + e^x| + c$

Case based Subjective Questions (2 mark each)

I. Read the following text and answer the following questions on the basis of the same:

Let's say that we want to evaluate $\int[P(x)/Q(x)]\, dx$, where $P(x)/Q(x)$ is a proper rational fraction. In such cases, it is possible to write the integrand as a sum of simpler rational functions by using partial fraction decomposition. Post this, integration can be carried out easily. The following image indicates some simple partial fractions which can be associated with various rational functions:

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$,
	where, x^2+bx+c cannot be factorised further	

In the above table, A, B and C are real numbers to be determined suitably.

Q. 1. Evaluate: $\int \frac{dx}{(x+1)(x+2)}$

Sol. We write,

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2} \quad ...(i)$$

where, real number A and B are to be determined suitably. This gives

$$1 = A(x+2)+B(x+1)$$

Equating the coefficients of x and the constant term, we get

$$A+B = 0$$

and $$2A+B = 1$$

Solving these equations, we get $A = 1$ and $B = -1$.

Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1}+\frac{-1}{x+2} \quad \mathbf{1}$$

Therefore,

$$\int\frac{dx}{(x+1)(x+2)} = \int\frac{dx}{x+1}-\int\frac{dx}{x+2}$$

$$= \log|x+1|-\log|x+2|+C$$

$$= \log\left|\frac{x+1}{x+2}\right|+C \quad \mathbf{1}$$

Q. 2. Evaluate: $\int \frac{1}{x^2-9}dx$

Sol. Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)}+\frac{B}{(x-3)}$

$$1 = A(x-3)+B(x+3)$$

Equating the coefficients of x and constant term, we obtain

$$A+B = 0$$

$$-3A+3B = 1$$

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\therefore \quad \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)}+\frac{1}{6(x-3)} \quad \mathbf{1}$$

$$\Rightarrow \quad \int\frac{1}{(x^2-9)}dx = \int\left(\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}\right)dx$$

$$= -\frac{1}{6}\log|x+3|+\frac{1}{6}\log|x-3| +C$$

$$= \frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right|+C \quad \mathbf{1}$$

Solutions for Practice Questions (Topic-1)

Very Short Answer Type Questions

2. Put $\sqrt{x} = t$

$$\therefore \int\frac{dx}{\sqrt{x}+x} = 2\log(1+\sqrt{x})+C \quad ½+½$$

[CBSE Marking Scheme 2020]

Detailed Solution:

$$\int\frac{dx}{\sqrt{x}+x}dx = \int\frac{1}{\sqrt{x}(1+\sqrt{x})}dx$$

Let, $1+\sqrt{x} = t$

$$0+\frac{1}{2\sqrt{x}}dx = dt$$

$$\frac{1}{\sqrt{x}}dx = 2dt$$

$$= 2\int\frac{1}{t}dt$$

$$= 2\log t + C$$

$$= 2\log(1+\sqrt{x}) + C$$

Commonly Made Error

- Mostly students find a difficulty in finding the correct substitution.

Answering Tip

- Practice more problems based on substitution.

9. $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dt = \int \left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} \right) dx$ ½

$= \int \tan x dx - \int \cot x dx$

$= \log|\sec x| - \log|\sin x| + C$

½

[CBSE Marking Scheme 2017]

Commonly Made Error

- Few candidates attempt it by integration by parts and make the steps complicated they could not proceed further.

Answering Tip

- This needs to be sufficiently practiced by students. Also, make students revise trigonometric and algebraic laws as well as basic integrals.

Detailed Solution :

Topper Answer, 2017

$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$

$= -2 \int \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} dx$

$= (-2) \int \frac{\cos 2x}{\sin 2x} dx$

$= (-2) \int \cot 2x \, dx$

$= (-2) \log|\sin 2x| \times \frac{1}{2} + C$ $\left[\int \cot x dx = \log|\sin x| \right]$

11. $\int \frac{3x}{3x-1} dx = \int \frac{3x-1+1}{3x-1} dx = \int \left(1 + \frac{1}{3x-1}\right) dx$

½

$= x + \frac{1}{3} \log|3x-1| + C$ ½

[CBSE Marking Scheme 2017]

Short Answer Type Questions-I

3.

$I = \int \frac{1}{\cos^2 x (1 - \tan x)^2} dx$

Put, $1 - \tan x = y$

So that, $-\sec^2 x \, dx = dy$ **1**

$= \int \frac{-1 dy}{y^2} = -\int y^{-2} dy$

$= \frac{1}{y} + c = \frac{1}{1 - \tan x} + c$

1

[CBSE SQP Marking Scheme 2020-21]

9. $I = \int \sin^{-1}(2x).1 dx$

$= x.\sin^{-1}(2x) - \int \frac{2x}{\sqrt{1-4x^2}} dx$ **1**

$= x.\sin^{-1}(2x) + \frac{1}{4} \int \frac{-8x}{\sqrt{1-4x^2}} dx$

$= x.\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + c$

1

[CBSE Marking Scheme, 2019]

OR

Topper Answer, 2017

$I = \int \underset{I}{1} \cdot \underset{II}{\sin^{-1}(2x)}\, dx$

$I = [\sin^{-1}(2x)](x) - \int \frac{(x)(2)}{\sqrt{1-4x^2}}\, dx$ [INTEGRATION by parts]

$I = x\sin^{-1}(2x) - \frac{1}{2}\int \frac{2x}{\sqrt{(\frac{1}{2})^2 - x^2}}\, dx$

$I = x\sin^{-1}(2x) + \frac{1}{4}\int \frac{(-8x)}{\sqrt{1-4x^2}}\, dx = x\sin^{-1}(2x) + \frac{1}{4} I_1$ where $I_1 = \int \frac{-8x}{\sqrt{1-4x^2}}\, dx$

Now, $I_1 = \int \frac{dt}{\sqrt{t}}$ let $1-4x^2 = t$, $-8x\,dx = dt$

$= 2\sqrt{t} + C$

$= 2\sqrt{1-4x^2} + C$

$I = x\sin^{-1}(2x) + \frac{1}{4}(2\sqrt{1-4x^2}) + C$

$I = x\sin^{-1}(2x) + \frac{\sqrt{1-4x^2}}{2} + C$

Commonly Made Error

- Mostly students have problems in integrating inverse trigonometric functions.

Answering Tip

- To integrate inverse trigonometric functions, apply by parts taking the second function as Ist Function.

14. Let $I = \sqrt{1 - \sin 2x}\, dx$

$= \int (\sin x - \cos x) dx$ **1**

as $\sin x > \cos x$ when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

$= -\cos x - \sin x + c$ **1**

[CBSE Marking Scheme, 2019]

Detailed Solution :

Let I $= \int \sqrt{1 - \sin 2x}\, dx, \frac{\pi}{4} < x < \frac{\pi}{2}$

$= \int \sqrt{(\sin^2 x + \cos^2 x) - 2\sin x \cos x}\, dx$

$= \int \sqrt{(\sin x - \cos x)^2}\, dx$

$= \int (\sin x - \cos x)\, dx$

$= \int \sin x\, dx - \int \cos x\, dx$

$= -\cos x - \sin x + c$

$= -(\cos x + \sin x) + c$

16. Put $\sin x = t$ or $\cos x\, dx = dt$

Integral reduces to $\int \frac{dt}{\sqrt{8 - t^2}} = \sin^{-1}\left(\frac{t}{\sqrt{8}}\right) + C$ **1**

$= \sin^{-1}\left(\frac{\sin x}{2\sqrt{2}}\right) + C$ **1**

[CBSE Marking Scheme 2017]

Short Answer Type Questions-II

3. $I = \int \frac{3x+5}{x^2+3x-18} dx = \frac{3}{2}\int \frac{2x+3}{x^2+3x-18} dx + \frac{1}{2}\int \frac{1}{x^2+3x-18} dx$

$$= \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx + \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}dx \quad \mathbf{1}$$

$$= \frac{3}{2}\log|x^2+3x-18| + \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + C \quad \mathbf{1+1}$$

[CBSE Marking Scheme, 2019]

10. Let $I = \int \frac{x}{(x^2+1)(x-1)}dx$

$\therefore \quad \frac{x}{(x^2+1)(x-1)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+1}$ ½

or $\quad x = A(x^2+1) + (x-1)(Bx+C)$ ½

or $\quad x = (A+B)x^2 + (C-B)x + (A-C)$

or $\quad A + B = 0,\ C - B = 1$ and $A - C = 0$

or $\quad A = \frac{1}{2}, B = -\frac{1}{2}$ and $C = \frac{1}{2}$ ½

$\therefore \quad I = \int \frac{\frac{1}{2}}{(x-1)}dx + \int \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}dx$ ½

$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + C$ **1**

[CBSE Marking Scheme 2017] (Modified)

Solutions for Practice Questions (Topic-2)

Very Short Answer Type Questions

2. $\int_0^a \frac{dx}{(2x)^2+1} = \frac{\pi}{8}$

$\Rightarrow \quad \frac{1}{2}[\tan^{-1}(2x)]_0^a = \frac{\pi}{8}$ ½

$\Rightarrow \quad a = \frac{1}{2}$ ½

[CBSE Marking Scheme 2020]

8. $\int_0^{2\pi} \cos^5 x\, dx = 2\int_0^{\pi} \cos^5 x\, dx$ ½

and $2\int_0^{\pi} \cos^5 x\, dx = 0$ or $\int_0^{2\pi} \cos^5 x\, dx = 0$ ½

[CBSE Marking Scheme, 2017]

Short Answer Type Questions-I

1. Put $2x = t$,

$\therefore \quad dx = \frac{1}{2}dt$ ½

$\therefore \quad I = \int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2}\right]e^{2x}dx$

$= \int_2^4 \left[\frac{1}{t} - \frac{1}{t^2}\right]e^t dt$ ½

$= \left[\frac{1}{t}e^t\right]_2^4 = \frac{e^4}{4} - \frac{e^2}{2}$ ½ + ½

[CBSE Marking Scheme 2020]

5. $I = \int_{-1}^{2} \frac{|x|}{x}dx = \int_{-1}^{0} -1dx + \int_0^2 1dx$ **1**

$= -1 + 2 = 1$ **1**

[CBSE Marking Scheme 2019]

Detailed Solution :

Let $\quad I = \int_{-1}^{2} \frac{|x|}{x}dx$

Since, $\quad \frac{|x|}{x} = \begin{cases} \frac{-x}{x}, & x < 0 \\ \frac{x}{x}, & x > 0 \end{cases}$

$= \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

$\therefore \quad I = \int_{-1}^{0}(-1)dx + \int_0^2 (1)dx$

$= [-x]_{-1}^0 + [x]_0^2$

$= -[0-(-1)] + (2-0)$

$= -1 + 2$

$= 1$

Short Answer Type Questions-II

1. $I = \int_0^1 \sqrt{3-2x-x^2}dx = \int_0^1 \sqrt{4-(x+1)^2}\,dx$ 1

Put $x + 1 = t \Rightarrow dx = dt$. when $x = 0, t = 0$ and when $x = 1, t = 2$ ½

$\therefore \quad I = \int_1^2 \sqrt{4-t^2}dt$

$= \left[\frac{t}{2}\sqrt{4-t^2} + \frac{4}{2}\sin^{-1}\left(\frac{t}{2}\right)\right]_1^2$ ½

$= \left[(0+2\sin^{-1}1) - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{1}{2}\right)\right]$ ½

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ ½

[CBSE Marking Scheme 2020] (Modified)

Detailed Solution:

$$\int_0^1 \sqrt{3-2x-x^2}\,dx = \int_0^1 \sqrt{-(x^2+2x-3)}\,dx$$

$$= \int_0^1 \sqrt{-(x^2+2x+1-4)}\,dx$$

$$= \int_0^1 \sqrt{(2)^2-(x+1)^2}\,dx$$

let $x + 1 = t$

$dx = dt$

when $x = 0, t = 1$

$x = 1, t = 2$

$$= \int_1^2 \sqrt{(2)^2-t^2}\,dt$$

[using $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$]

$$= \left[\frac{t}{2}\sqrt{(2)^2-t^2}+\frac{(2)^2}{2}\sin^{-1}\frac{t}{2}\right]_1^2$$

$$= \left[\frac{2}{2}\sqrt{4-4}+2\sin^{-1}\frac{2}{2}\right]-\left[\frac{1}{2}\sqrt{4-1}+2\sin^{-1}\frac{1}{2}\right]$$

$$= 0+2\left(\frac{\pi}{2}\right)-\left[\frac{\sqrt{3}}{2}+2\left(\frac{\pi}{6}\right)\right]$$

$$= \pi-\frac{\sqrt{3}}{2}-\frac{\pi}{3}$$

$$= \frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

$$= \frac{4\pi-3\sqrt{3}}{6}$$

2. Let $I = \int_0^a f(a-x)\,dx$

Put $a - x = t \Rightarrow -dx = dt$ ½

$I = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$ ½

II part.

$$I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi-x)\cdot\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow 2I = \int_0^\pi \frac{\pi\cdot\sin x}{1+\cos^2 x}dx$$ 1

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\Rightarrow I = -\frac{\pi}{2}\cdot\int_1^{-1}\frac{dt}{1+t^2} = \frac{\pi}{2}\times 2\times\int_0^1\frac{dt}{1+t^2}$$

$$= \pi\left[\tan^{-1}t\right]_0^1 = \frac{\pi^2}{4}$$ 1

[CBSE Marking Scheme, 2019] (Modified)

Long Answer Type Questions-I

5. $I = \int_0^{\frac{\pi}{4}} \log[1+\tan x]\,dx$ ...(i)

Apply the property $\int_0^a f(x) = \int_0^a f(a-x)$

$$I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

or $I = \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan x}\right]dx$ 1

or $I = \int_0^{\frac{\pi}{4}} [\log 2-\log(1+\tan x)]\,dx$ ...(ii) 1

Adding eqn. (i) and (ii), we get

or $2I = \int_0^{\frac{\pi}{4}} \log 2\,dx = \frac{\pi}{4}\log 2$

or $I = \frac{\pi}{8}\log 2.$ 1

[CBSE Marking Scheme 2015] (Modified)

Long Answer Type Questions-II

1. Put $\sin x - \cos x = t,$ 1

$(\cos x + \sin x)\,dx = dt.$

$1 - \sin 2x = t^2$

when $x = 0, \quad t = -1$

and $x = \frac{\pi}{4}, \quad t = 0$ ½

$$\therefore \quad I = \int_0^{\pi/4} \frac{\sin x+\cos x}{16+9\sin 2x}dx$$

$$= \int_{-1}^0 \frac{1}{16+9(1-t^2)}dt$$

$$= \int_{-1}^0 \frac{1}{25-9t^2}dt$$ 1

$$\Rightarrow \quad I = \frac{1}{30}\log\left|\frac{5+3t}{5-3t}\right|_{-1}^0$$ 1½

$$= \frac{1}{30}\left[0 \quad \log\frac{1}{4}\right]$$

$$= -\frac{1}{30}\log\frac{1}{4} \text{ or } \frac{1}{15}\log 2$$ 1

[CBSE Marking Scheme 2018] (Modified)

REFLECTION

- Integration is mainly used to find the areas of the two dimensional region and computing volumes of three dimensional objects.
- Finding a integral of a function w.r.t. x means finding the area to the X-axis from the curve.
- Integrals are used to calculate the trajectory (path) and the velocity of a satellite at the time of placing it in orbit.